# Solving the multiplication constraint in several approximation spaces

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**Contents**

# Introducing the problem

# 1. Starting examples

- $$\begin{pmatrix} z = xy \\ \wedge \;\; x \in [-2, 5] \\ \wedge \;\; y \in [-3, 1] \\ \wedge \;\; z \in [-14, 17] \end{pmatrix} \leftrightarrow \begin{pmatrix} z = xy \\ \wedge \;\; x \in [-2, 5] \\ \wedge \;\; y \in [-3, 1] \\ \wedge \;\; z \in [-14, 6] \end{pmatrix}$$

- $$\begin{pmatrix} z = xy \\ \wedge \;\; x \in [5, 6] \\ \wedge \;\; y \in [-4, 2] \\ \wedge \;\; z \in [-15, 13] \end{pmatrix} \leftrightarrow \begin{pmatrix} z = xy \\ \wedge \;\; x \in [5, 6] \\ \wedge \;\; y \in [-3, 2] \\ \wedge \;\; z \in [-15, 12] \end{pmatrix}$$

- $$\begin{pmatrix} z = xy \\ \wedge \;\; x \in [1, 9] \\ \wedge \;\; y \in [2, 4] \\ \wedge \;\; z \in [8, 16] \end{pmatrix} \leftrightarrow \begin{pmatrix} z = xy \\ \wedge \;\; x \in [2, 8] \\ \wedge \;\; y \in [2, 4] \\ \wedge \;\; z \in [8, 16] \end{pmatrix}$$

- $$\begin{pmatrix} z = xy \\ \wedge \;\; x \in [-1, 1] \\ \wedge \;\; y \in [-1, 1] \\ \wedge \;\; z \in [2, 2] \end{pmatrix} \Longrightarrow \begin{pmatrix} z = xy \\ \wedge \;\; x \in \emptyset \\ \wedge \;\; y \in \emptyset \\ \wedge \;\; z \in \emptyset \end{pmatrix}$$

# 2. A view of the multiplication

- *times* $:= \{(x, y, z) \in \mathbf{R}^3 \mid z = xy\}$

# 3. Another view of the multiplication

- *times* $:= \{(x,y,z) \in \mathbf{R}^3 \mid z = xy\}$

# 4. Several kinds of intervals

• Let $(\mathbf{D}, \preceq)$ be a totally ordered set.

• **Definition** A (true) *interval* is a possibly empty subset of **D**, of the form

$$\{x \in \mathbf{D} \,|\, e \preceq x \text{ and } x \preceq d\}$$

with $e, d$ any elements of **D**. It is written

$$[e, d]$$

• **Definition** A *convex* subset of **D** is a subset $a$ of **D** such that, for all $x \in \mathbf{D}$ and $y \in \mathbf{D}$,

$$x \in a \text{ and } y \in a \;\rightarrow\; [x, y] \subseteq a.$$

# 5. Several kinds of intervals, next

• **Property** The set of intervals of $(\mathbf{R}, \leq)$ is closed by finite intersection but not by infinite intersection.

• **Property** The set of convex subsets of $(\mathbf{R}, \leq)$ is closed by infinite intersection and is equal to the set of *generalized intervals* of **R**, that is to say, the set of subsets of **R** of one of the 10 forms:

| | | | |
|---|---|---|---|
| 1. | $\emptyset$, | | |
| 2. | $\{x \in \mathbf{R} \mid e \leq x$ and $x \leq d\}$, | written | $[e, d]$, |
| 3. | $\{x \in \mathbf{R} \mid e \leq x$ and $x < d\}$, | written | $[e, d)$, |
| 4. | $\{x \in \mathbf{R} \mid e < x$ and $x \leq d\}$, | written | $(e, d]$, |
| 5. | $\{x \in \mathbf{R} \mid e < x$ and $x < d\}$, | written | $(e, d)$, |
| 6. | $\{x \in \mathbf{R} \mid e \leq x\}$, | written | $[e, +\infty)$, |
| 7. | $\{x \in \mathbf{R} \mid x \leq d\}$, | written | $(-\infty, d]$, |
| 8. | $\{x \in \mathbf{R} \mid e < x\}$, | written | $(e, +\infty)$, |
| 9. | $\{x \in \mathbf{R} \mid x < d\}$, | written | $(-\infty, d)$, |
| 10. | **R**, | written | $(-\infty, +\infty)$, |

where $e, d$ are elements of **R**, with $e < d$, but case 2, where $e \leq d$.

# 6. The general constraint solving problem

• Let **D** be a set and $\mathcal{D}$ a set of *selected* subsets of **D**. A $n$-*bloc* is a Cartesian product of $n$ selected subsets.

• Given a subset $r$ of $\mathbf{D}^n$ and elements $a_1, \ldots, a_n$ of $\mathcal{D}$, we are interested in computing, if they exists, elements $a'_1, \ldots, a'_n$ of $\mathcal{D}$ such that $a'_1 \times \cdots \times a'_n$ is the least $n$-bloc for which the folowing equivalence of constraints holds:

$$\begin{pmatrix} & (x_1, \ldots, x_n) \in r \\ \wedge & x_1 \in a_1 \\ & \ldots \\ \wedge & x_n \in a_n \end{pmatrix} \leftrightarrow \begin{pmatrix} & (x_1, \ldots, x_n) \in r \\ \wedge & x_1 \in a'_1 \\ & \ldots \\ \wedge & x_n \in a'_n \end{pmatrix}$$

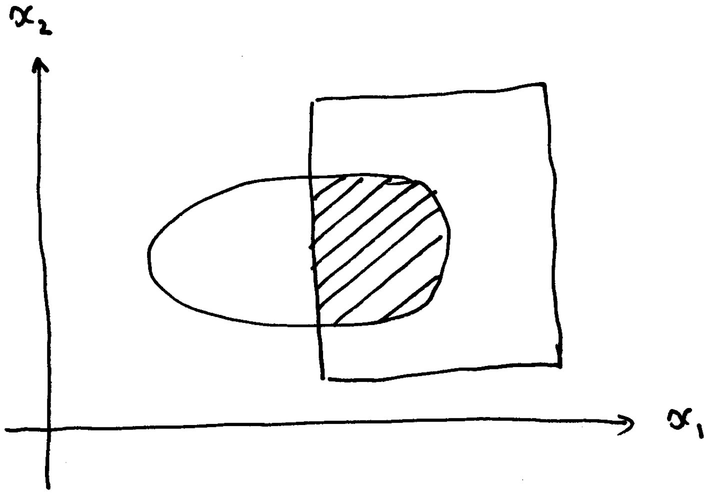
• Thus $a'_1 \times \cdots \times a'_n$ is the least $n$-bloc such that

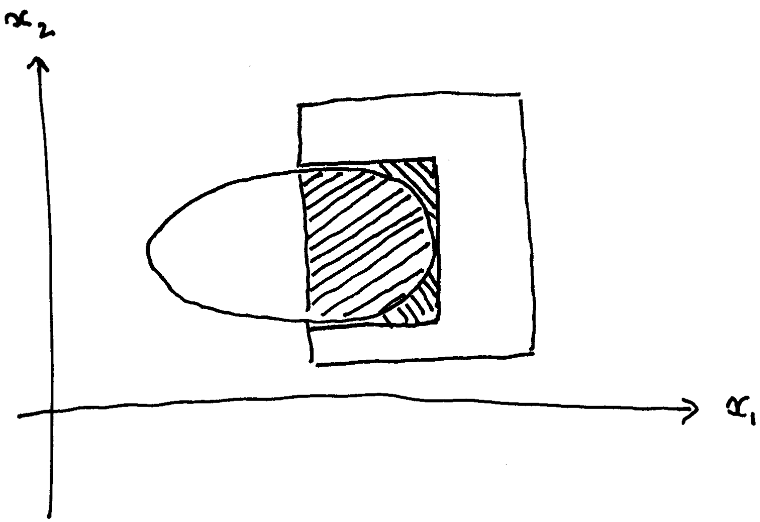
$$r \ \cap \ a_1 \times \cdots \times a_n \ = \ r \ \cap \ a'_1 \times \cdots \times a'_n,$$

• which is also the least $n$-bloc such that

$$r \ \cap \ a_1 \times \cdots \times a_n \ \subseteq \ a'_1 \times \cdots \times a'_n.$$

# 7. Visualization of the problem



# 8. Approximation space

• **Definition** An *approximation space* is an ordered pair $(\mathbf{D}, \mathcal{D})$, where

1. **D** is any set,
2. $\mathcal{D}$ is a set of *selected* subsets of **D**,
3. **D** and $\emptyset$ belong to $\mathcal{D}$.

The space is *total* if for any subset $r$ of **D** the subset of **D**, denoted and defined by,

$$\mathit{apx}_{\mathcal{D}}(r) \;:=\; \bigcap\{x \in \mathcal{D} \,|\, r \subseteq x\},$$

belongs to $\mathcal{D}$.

# 9. Approximation space, next

• **Remark 1** In the partially ordered set $(\mathcal{P}(\mathbf{D}), \subseteq)$, the set *apx* $_{\mathcal{D}}(r)$ is the greatest lower bound of the set of elements of $\mathcal{D}$ which contain $r$:

$$\mathit{apx}_{\mathcal{D}}(r) \;:=\; \inf_{\mathcal{D}} \{x \in \mathcal{D} \,|\, r \subseteq x\}$$

• **Remark 2** If $(\mathbf{D}, \mathcal{D})$ is total then, in the partially ordered set $(\mathcal{P}(\mathbf{D}), \subseteq)$, the set *apx* $_{\mathcal{D}}(r)$ is always the greatest element of the set of elements of $\mathcal{D}$ which contain $r$:

$$\mathit{apx}_{\mathcal{D}}(r) \;:=\; \min \{x \in \mathcal{D} \,|\, r \subseteq x\}$$

• **Remark 3** A sufficient condition in order that $(\mathbf{D}, \mathcal{D})$ is total, is that $\mathcal{D}$ is closed by intersection (finite and infinite)

# 10. Properties of *apx*

- Let $(\mathbf{D}, \mathcal{D})$ be an approximation space. We write *apx* for *apx* $_{\mathcal{D}}$.
- **Properties** For any subsets $r, s$ of **D** and any family of subsets $r_i$ of **D**, indexed by a set $I$:

$$
\begin{array}{lll}
\text{(i)} & r \subseteq \mathit{apx}\,(r), & \text{(contraction)},\\
\text{(ii)} & \mathit{apx}\,(\mathit{apx}\,(r)) = \mathit{apx}\,(r), & \text{(idempotence)},\\
\text{(iii)} & r \subseteq s \quad \rightarrow \quad \mathit{apx}\,(r) \subseteq \mathit{apx}\,(s), & \text{(weakly increasing)},\\
\text{(iv)} & \mathit{apx}\,(\cup_{i \in I} r_i) = \mathit{apx}\,(\cup_{i \in I} \mathit{apx}\,(r_i)). &
\end{array}
$$

- Let $(\mathbf{D}^n, \mathcal{D}^{(n)})$ be the approximation space defined by

$$\mathcal{D}^{(n)} \;:=\; \{a_1 \times \cdots \times a_n \mid \; a_i \in \mathcal{D}, \text{ for all } i \in 1..n\},$$

The elements of $\mathcal{D}^{(n)}$ are called $n$-*blocs* on $\mathcal{D}$.

- **Property** If $r$ is a subset of $\mathbf{D}^n$ then,

$$\mathit{apx}_{\mathcal{D}^{(n)}}(r) \;=\; \mathit{apx}_{\mathcal{D}}(\pi_1(r)) \times \cdots \times \mathit{apx}_{\mathcal{D}}(\pi_n(r)).$$

where the $i$-*th projection* of $r$ is denoted and defined by

$$\pi_i(r) := \{d \in \mathbf{D} \,|\, \text{there exists } (d_1, \ldots, d_n) \in r \text{ with } d = d_i\}.$$

- **Conclusion** Given a subset $r$ of $\mathbf{D}^n$ and a $n$-bloc $a$ on $\mathcal{D}$, we want to compute

$$\mathit{apx}_{\mathcal{D}^{(n)}}(r \cap a)$$

## 11. Our approximation spaces

• Let $(\mathbf{R}, \leq)$ be the ordered set of real numbers and let $\mathbf{F}$ be a finite subset of $\mathbf{R}$ of the form

$$\mathbf{F} = \{f_{-k}, f_{-k+1}, \ldots, f_1, f_0, f_1, \ldots, f_{k-1}, f_k\},$$

with $f_i < f_{i+1}$ and $f_{-i} = -f_i$

• We consider the four approximation spaces of the form

$$(\mathbf{R}^3, \mathcal{R}^{(3)}),$$

with $\mathcal{R}$ being successively

1. the set composed of $\mathbf{R}$ and the (true) intervals of $(\mathbf{R}, \leq)$,
2. the set of convex subsets of $(\mathbf{R}, \leq)$,
3. the set composed of $\mathbf{R}$, $\emptyset$ and the *machine* intervals of $(\mathbf{R}, \leq)$, that is to say, the non-empty intervals whose endpoints belong to $\mathbf{F}$,
4. the set of convex *machine* subsets of $(\mathbf{R}, \leq)$, that is to say, those intervals whose greatest lower bounds and least upper bounds, if they exist, belong to $\mathbf{F}$.

• For each of these spaces, given $a \in \mathcal{R}^{(3)}$, we want to compute

$$\mathit{apx}_{\mathcal{R}^{(3)}}(\mathit{times} \cap a)$$

with

$$\mathit{times} := \{(x_1, x_2, x_3) \in \mathbf{R}^3 \,|\, x_3 = x_1 x_2\}.$$

# 12. Sharper approximation space

• The following property allow us to replace the *apx* computations in the approximation spaces $1$ and $3$ by *apx* computations in the approximation spaces $2$ and $4$.

• **Property** If $(\mathbf{D}, \mathcal{D})$ and $(\mathbf{D}', \mathcal{D}')$ are approximation spaces such that

$$\mathbf{D} \subseteq \mathbf{D}', \;\; \mathcal{D} \subseteq \mathcal{D}'$$

then, for any $r \subseteq \mathbf{D}$,

$$\mathit{apx}_{\mathcal{D}}(r) = \mathit{apx}_{\mathcal{D}}(\mathit{apx}_{\mathcal{D}'}(r))$$

# 13. Sharper approximation space, next

- **Proof** We must prove that

$$\bigcap\{a \in \mathcal{D} \,|\, r \subseteq a\} \;=\; \bigcap\{a \in \mathcal{D} \,|\, \mathit{apx}_{\mathcal{D}'}(r) \subseteq a\}$$

Thus it is sufficient to prove that for all $a \in \mathcal{D}$ we have

$$r \subseteq a \;\leftrightarrow\; \mathit{apx}_{\mathcal{D}'}(r) \subseteq a$$

If $r \subseteq a$ then $\mathit{apx}_{\mathcal{D}'}(r) \subseteq \mathit{apx}_{\mathcal{D}'}(a)$ and, since $a \in \mathcal{D}'$, we have $\mathit{apx}_{\mathcal{D}'}(r) \subseteq a$. Thus

$$r \subseteq a \;\rightarrow\; \mathit{apx}_{\mathcal{D}'}(r) \subseteq a$$

Since $r \subseteq \mathit{apx}_{\mathcal{D}'}(r)$,

$$\mathit{apx}_{\mathcal{D}'}(r) \subseteq a \;\rightarrow\; r \subseteq a$$

# 14. Translation of an approximation space

• The computations of *apx* $(\textit{times} \cap a)$ in the approximation spaces $2$ and $4$ will be performed by translations and computations into more convenient approximation spaces.

• **Definition** A *translation* of a total approximation space $(\mathbf{D}, \mathcal{D})$ into an approximation space $(\mathbf{D}', \mathcal{D}')$ is a mapping $\varphi$, of type $\mathcal{P}(\mathbf{D}) \to \mathcal{P}(\mathbf{D}')$, such that,

1. $\varphi(\textit{apx}_{\mathcal{D}}(r \cap a)) = \textit{apx}_{\mathcal{D}'}(\varphi(r) \cap \varphi(a))$, for all $r \subseteq \mathbf{D}$ and $a \in \mathcal{D}$,
2. the restriction of $\varphi$ to $\mathcal{D}$ defines an injective mapping of type $\mathcal{D} \to \mathcal{D}'$.

# 15. Convenient approximation space

• Let $(\mathbf{D}, \preceq)$ be a totally ordered set having a least element $\min(\mathbf{D})$ and a greatest element $\max(\mathbf{D})$ and let $\mathcal{D}$ be the set of intervals of $\mathbf{D}$.

• **Property** For all intervals $a_i$, even empty,

$$a_1 \cap \cdots \cap a_n = \begin{bmatrix} \max\{\inf(a_1), \ldots, \inf(a_n)\}, \\ \min\{\sup(a_1), \ldots, \sup(a_n)\} \end{bmatrix}.$$

• **Property** In the approximation space $(\mathbf{D}, \mathcal{D})$, for all intervals $a_i$, even empty,

$$\mathit{apx}\,(a_1 \cup \cdots \cup a_n) = \begin{bmatrix} \min\{\inf(a_1), \ldots, \inf(a_n)\}, \\ \max\{\sup(a_1), \ldots, \sup(a_n)\} \end{bmatrix}.$$

# About ordered sets: computing minima and maxima

# 16. Multi-monotonic function

• Let $a$ be a $m$-box in $(\mathbf{D}, \preceq)$, that is to say, a Cartesian product of $m$ intervals of $(\mathbf{D}, \preceq)$.

• **Definition** *The set of endings* of $a$ is denoted and defined by

$$\textit{ends}\,(a) := \begin{cases} \emptyset, \text{ if } a = \emptyset, \text{ else} \\ \{\min(\pi_1(a)), \max(\pi_1(a))\} \times \cdots \times \{\min(\pi_m(a)), \max(\pi_m(a))\} \end{cases}$$

• **Definition** A function of type $a \to \mathbf{D}$ is *multi-monotonic*, if for all $(\alpha_1, \ldots, \alpha_m) \in a$ and all $i \in 1..m$, the mapping

$$x \mapsto f(\alpha_1, \ldots, \alpha_{i-1}, x, \alpha_{i+1}, \ldots, \alpha_m),$$

of type $\pi_i(a) \to \mathbf{D}$, is monotonic, that is to say, weakly increasing or weakly decreasing.

• **Example** of a multi-monotonic function in $(\mathbf{R}, \leq)$, with $a = [-2, 5] \times [-3, 1]$,

$$f : (x_1, x_2) \mapsto x_1 x_2.$$

• **Property of the endings** Let $a$ be a non empty $m$-box in $(\mathbf{D}, \preceq)$ and $f$ a multi-monotonic function of type $a \to \mathbf{D}$. The elements

$$\min\{f(x) \mid x \in a\}, \ \max\{f(x) \mid x \in a\}$$

exist and are respectivly equal to

$$\min\{f(x) \mid x \in \textit{ends}\,(a)\}, \ \max\{f(x) \mid x \in \textit{ends}\,(a)\}.$$

# 17. Computing a maximum, example

- In $(\mathbf{R}, \leq)$, we want to compute

$$y \;=\; \max\{x_1 x_2 \,|\, x_1 \in [-2,5] \text{ and } x_2 \in [-3,1]\}$$

- Since multiplication is multi-monotonic,

$$y \;=\; \max\{(-2)(-3), (-2)(+1), (+5)(-3), (+5)(+1)\} \;=\; \max\{6, -2, -15, 5\} \;=\; 6$$

# About ordered sets: convex projections

# 18. Moving from dimension $n$ to dimension 2

• Let $(\mathbf{D}, \preceq)$ be an ordered set.

• **Definition** The subset $r$ of $\mathbf{D}^n$, with $n \geq 2$, *preserves convexity* if, for all convex subsets $a_i$ of $(\mathbf{D}, \preceq)$, the projection $\pi_n(r \cap a_1 \times \cdots \times a_{n-1} \times \mathbf{D})$ is convex.

• **Definition** Let $r$ be a subset of $\mathbf{D}^n$, with $n \geq 2$. A $i$*-cut* of $r$, is a subset of $\mathbf{D}^2$ of the form

$$\{(x_1, x_2) \in \mathbf{D}^2 \mid (\alpha_1, \ldots, \alpha_{i-1}, x_1, \alpha_{i+1}, \ldots, \alpha_{i-1}, x_2) \in r\},$$

with each $\alpha_j \in \pi_j(r)$ and $i$ taken between $1$ and $n-1$.

• **Theorem** A sufficient condition in order that a subset of $\mathbf{D}^n$, with $n \geq 3$, preserves convexity, is that each $i$-cut $s$ of $r$ preserves convexity and is such that $\pi_1(s) = \pi_i(r)$.

# 19. The dimension 2 case

**Theorem** A sufficient condition in order that a subset $r$ of $\mathbf{D}^2$ preserves convexity, is that

- the projection $\pi_2(r)$ is convex,
- when $r$ is not empty, there exist monotonic mappings $\underline{r}$ and $\overline{r}$, of type $\pi_1(r) \longrightarrow \mathbf{D}$ such that,

$$(x_1, x_2) \in r \;\leftrightarrow\; x_2 \in [\underline{r}(x_1),\, \overline{r}(x_2)],$$

  for all $x_1 \in \pi_1(r)$ and $x_2 \in \mathbf{D}$.

# 20. Example of a relation which preserves convexity

• Consider the subset of $\mathbf{R}^3$.

$$\mathit{corners} := \begin{pmatrix} \{(x,y,z) \in [-1,1]\times[-1,0]\times\mathbf{R} \mid (x+1)^2 + z^2 \leq 4\} \cup \\ \{(x,y,z) \in [-1,1]\times(0,1]\times\mathbf{R} \mid (x-1)^2 + z^2 \leq 4\} \end{pmatrix}$$

- This subset preserves convexity, since its $i$-cuts are of one of the 4 forms:

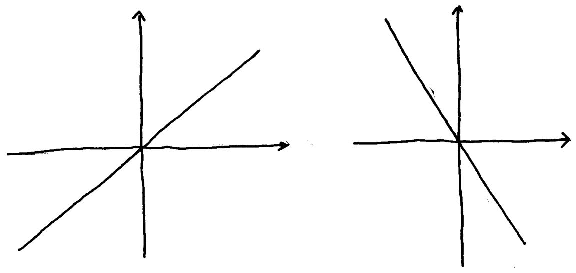
# 21. Other examples of relations which preserve convexity

• The following relations preserve convexity

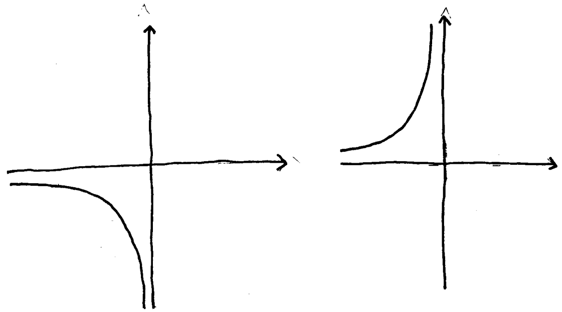
$$\begin{array}{lcl} \mathit{times} & := & \{(x,y,z) \in \mathbf{R}^3 \mid z = xy\}, \\ \mathit{ndivision} & := & \{(x,y,z) \in \mathbf{R}^3 \mid x = yz \text{ and } y < 0\}, \\ \mathit{zdivision} & := & \{(x,y,z) \in \mathbf{R}^3 \mid x = yz \text{ and } y = 0\}, \\ \mathit{pdivision} & := & \{(x,y,z) \in \mathbf{R}^3 \mid x = yz \text{ and } y > 0\}, \end{array}$$

• because *zdivision* $= (\{0\} \times \{0\} \times \mathbf{R}) \cup (\{\mathbf{R}\} \times \{0\} \times \{0\})$,

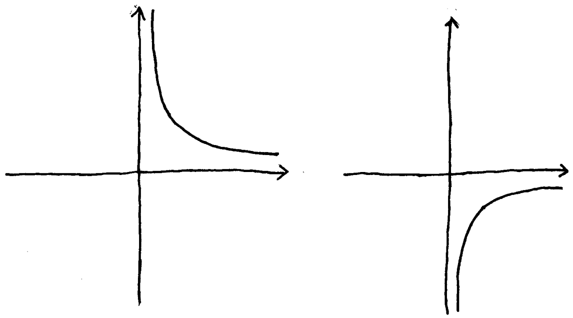
• the $1$-cuts and $2$-cuts of *times* and the $1$-cuts of *ndivision* and *pdivision* are of one of the two forms,



• the $2$-cuts of *ndivision* are of one of the two forms,



• and the $2$-cuts of *pdivision* are of one of the two forms,



# 22. Decomposition of the multiplication

- Let $a$ be a Cartesain product of convex subsets of **R**.

- We have

$$
\begin{aligned}
&(\pi_1(\mathit{times} \cap a),\ \pi_2(\mathit{times} \cap a),\ \pi_3(\mathit{times} \cap a)) \ = \\
&\qquad \left( \begin{array}{l}
\left( \begin{array}{l} a_1 \cap \pi_3(\mathit{ndivision} \cap\ a_3 \times a_2 \times \mathbf{D}) \cup \\ a_1 \cap \pi_3(\mathit{zdivision} \cap\ a_3 \times a_2 \times \mathbf{D}) \cup \\ a_1 \cap \pi_3(\mathit{pdivision} \cap\ a_3 \times a_2 \times \mathbf{D}) \end{array} \right), \\
\\
\left( \begin{array}{l} a_2 \cap \pi_3(\mathit{ndivision} \cap\ a_3 \times a_1 \times \mathbf{D}) \cup \\ a_2 \cap \pi_3(\mathit{zdivision} \cap\ a_3 \times a_1 \times \mathbf{D}) \cup \\ a_2 \cap \pi_3(\mathit{pdivision} \cap\ a_3 \times a_1 \times \mathbf{D}) \end{array} \right), \\
\\
a_3 \cap \pi_3(\mathit{times} \cap\ a_1 \times a_2 \times \mathbf{D})
\end{array} \right),
\end{aligned}
$$

$$
\begin{aligned}
a_i &:= \pi_i(a)), \\
\mathit{ndivision} &:= \{(u,v,w) \in \mathbf{R}^3 \mid (w,v,u) \in \mathit{times} \text{ and } v < 0\}, \\
\mathit{zdivision} &:= \{(u,v,w) \in \mathbf{R}^3 \mid (w,v,u) \in \mathit{times} \text{ and } v = 0\}, \\
\mathit{pdivision} &:= \{(u,v,w) \in \mathbf{R}^3 \mid (w,v,u) \in \mathit{times} \text{ and } v > 0\}.
\end{aligned}
$$

• and each set of one of the forms

$$a_k \cap \pi_3(\textit{times} \cap\ a_i \times a_j \times \mathbf{D}),$$
$$a_k \cap \pi_3(\textit{ndivision} \cap\ a_i \times a_j \times \mathbf{D}),$$
$$a_k \cap \pi_3(\textit{zdivision} \cap\ a_i \times a_j \times \mathbf{D}),$$
$$a_k \cap \pi_3(\textit{pdivision} \cap\ a_i \times a_j \times \mathbf{D}),$$

is convex.

# About orderded sets: computing projections

# 23. Good relation

• **Definition** A *good* $n$-ary relation, with $n \geq 3$, is a subset $r$ of $\mathbf{D}^n$ whose $i$-cuts $s$ are all good binary relations and are such that $\pi_1(s) = \pi_i(r)$.

• **Definition** A *good* binary relation is a subset $r$ of $\mathbf{D}^2$ such that

- the projection $\pi_1(r)$ is convex and admits a least element if it is lower bounded, and a greatest element, if it is upper bounded,
- the projection $\pi_2(r)$ is convex,
- when $r$ is not empty, there exist monotonic mappings $\underline{r}$ and $\overline{r}$, of type $\pi_1(r) \to \mathbf{D}$ such that,

$$(x_1, x_2) \in r \;\leftrightarrow\; x_2 \in [\underline{r}(x_1),\, \overline{r}(x_2)],$$

  for all $x_1 \in \pi_1(r)$ and $x_2 \in \mathbf{D}$.

# 24. Computation of projections

• **Theorem** For any good $n$-ary relation and $n$-box $a$ in $(\mathbf{D}, \preceq)$, with $n \geq 2$,

$$\pi_n(r \cap a) = \begin{cases} \emptyset, \text{ if } \Delta = \emptyset, \text{ else} \\ \pi_n(a) \cap \begin{bmatrix} \min\{\underline{r}(x) \,|\, x \in \textit{ends}\,(\Delta)\}, \\ \max\{\overline{r}(x) \,|\, x \in \textit{ends}\,(\Delta)\} \end{bmatrix} \end{cases},$$

with

$$\begin{aligned} \Delta &:= (\pi_1(r) \cap \pi_1(a)) \times \cdots \times (\pi_{n-1}(r) \cap \pi_{n-1}(a)), \\ \underline{r}(x_1, \ldots, x_{n-1}) &:= \min(\{x_n \,|\, (x_1, \ldots, x_n) \in r\}), \\ \overline{r}(x_1, \ldots, x_{n-1}) &:= \max(\{x_n \,|\, (x_1, \ldots, x_n) \in r\}), \end{aligned}$$

the functions $\underline{r}, \overline{r}$ being necessarily defined when the box $\Delta$ is not empty.

• **Remark** If $(\mathbf{D}, \preceq)$ admits a least and a greatest element, then the first formula of the theorem simplifies to

$$\pi_n(r \cap a) = \pi_n(a) \cap \begin{bmatrix} \inf\{\underline{r}_n(x) \,|\, x \in \textit{ends}\,(\Delta)\}, \\ \sup\{\overline{r}_n(x) \,|\, x \in \textit{ends}\,(\Delta)\} \end{bmatrix}.$$

# 25. Projection of the double corner

• We want to compute $\pi_3(\mathit{corners} \cap a)$ with $a$ a $3$-box of $(\mathbf{R}, \leq)$ and

$$\mathit{corners} := \begin{pmatrix} \{(x,y,z) \in [-1,1] \times [-1,0] \times \mathbf{R} \mid (x+1)^2 + z^2 \leq 4\} \cup \\ \{(x,y,z) \in [-1,1] \times (0,1] \times \mathbf{R} \mid (x-1)^2 + z^2 \leq 4\} \end{pmatrix}$$

• Thus

$$\pi_3(\mathit{corners} \cap a) = \begin{cases} \emptyset, \text{ if } \Delta = \emptyset, \text{ else} \\ \pi_3(a) \cap \begin{bmatrix} \min\{ \underline{\mathit{corners}}(u) \mid u \in \mathit{ends}(\Delta)\}, \\ \max\{\overline{\mathit{corners}}(u) \mid u \in \mathit{ends}(\Delta)\} \end{bmatrix} \end{cases}$$

with

$$\Delta := (\pi_1(a) \cap [-1,1]) \times (\pi_2(a) \cap [-1,1]),$$

$$\underline{\mathit{corners}}(x,y) := 0,$$

$$\overline{\mathit{corners}}(x,y) = \begin{cases} \sqrt{4-(x+1)^2}, & \text{if } y \leq 0, \\ \sqrt{4-(x-1)^2}, & \text{if } y > 0. \end{cases}$$

- Finally

$$\pi_3(\textit{corners} \cap a) =$$

$$\begin{cases} \emptyset, \text{ if } a_1 = \emptyset \lor a_2 = \emptyset, \text{ else} \\ a_3 \cap \begin{cases} [0, \sqrt{4 - (\underline{a_1} + 1)^2}], & \text{if } \overline{a_2} \le 0 \lor (\underline{a_2} \le 0 \land -\underline{a_1} \ge \overline{a_1}), \\ [0, \sqrt{4 - (\overline{a_1} - 1)^2}], & \text{if } \underline{a_2} > 0 \lor (\overline{a_2} > 0 \land -\underline{a_1} < \overline{a_1}) \end{cases} \end{cases}$$

$$\begin{aligned} a_1 &:= \pi_1(a) \cap [-1, 1], \\ a_2 &:= \pi_2(a) \cap [-1, 1], \\ a_3 &:= \pi_3(a), \\ \underline{a_i} &:= \min(a_i), \quad \text{when } a_i \neq \emptyset, \\ \overline{a_i} &:= \max(a_i), \quad \text{when } a_i \neq \emptyset. \end{aligned}$$

# Aggregated approximation space

# 26. Aggregated approximation space

• **Definition** Given a total approximation space $(D, \mathcal{D})$ and a subset $r$ of **D** we denote and define the *aggregate* of $r$ by

$$\textit{agr}_{\mathcal{D}}(r) := \{\textit{apx}_{\mathcal{D}}(\{x\}) \,|\, x \in r\}$$

• **Definition** The *agregate* of the approximation space $(D, \mathcal{D})$ is then the approximation space

$$(D', \mathcal{D}') := (\textit{agr}_{\mathcal{D}}(D),\ \{\textit{agr}_{\mathcal{D}}(a) \,|\, a \in \mathcal{D}\}$$

The elements of $\textit{agr}_{\mathcal{D}}(D)$ are called *atomic elements* of $\mathcal{D}$.

• **Remark** If $(\mathbf{D}', \mathcal{D}')$ is the aggregate space of $(\mathbf{D}, \mathcal{D})$ then $(\mathbf{D}'^n, \mathcal{D}'^{(n)})$ is the aggregate space of $(\mathbf{D}^n, \mathcal{D}^{(n)})$ and

$$\textit{agr}_{\mathcal{D}^{(n)}}(r_1 \times \cdots \times r_n) \ = \ \textit{agr}_{\mathcal{D}}(r_1) \times \cdots \times \textit{agr}_{\mathcal{D}}(r_n).$$

# 27. Example

• Let $(\mathbf{R}, \mathcal{R})$ be the approximation space, where $\mathcal{R}$ is the set of convex subsets of **R**, whose lower an upper bounds, if they exist, belong to

$$\mathbf{F} := \{-1, 0, 1\}.$$

The aggregate space of $(\mathbf{R}, \mathcal{R})$ is $(\mathbf{R}', \mathcal{R}')$,

• where

$$\mathbf{R}' := \left\{ \begin{matrix} \langle -3 \rangle, \\ \langle -2 \rangle, \\ \langle -1 \rangle, \\ \langle 0 \rangle, \\ \langle 1 \rangle, \\ \langle 2 \rangle, \\ \langle 3 \rangle \end{matrix} \right\} := \left\{ \begin{matrix} (-\infty, -1), \\ [-1, -1], \\ (-1, 0), \\ [0, 0], \\ (0, 1), \\ [1, 1], \\ (1, +\infty) \end{matrix} \right\},$$

• and $\mathcal{R}'$ is the set of intervals of $(\mathbf{D}', \preceq)$, with $\langle i \rangle \preceq \langle j \rangle \;\leftrightarrow\; i \leq j$.

# 28. Example, next

• Consider the subset of $\mathbf{R}^2$

$$\textit{inverse} := \{(x, y) \in \mathbf{R}^2 \mid xy = 1\}.$$

• According to

• we get

$$agr_{\mathcal{R}^{(2)}}(\mathit{inverse}) = \begin{Bmatrix} (-\infty,-1)\times(-1,0), \\ [-1,-1]\times[-1,-1], \\ (-1,0)\times(-\infty,-1), \\ (0,1)\times(1,+\infty), \\ [1,1]\times[1,1], \\ (1,+\infty)\times(0,1) \end{Bmatrix} = \begin{Bmatrix} \langle -3\rangle\times\langle -1\rangle, \\ \langle -2\rangle\times\langle -2\rangle, \\ \langle -1\rangle\times\langle -3\rangle, \\ \langle 1\rangle\times\langle 3\rangle, \\ \langle 2\rangle\times\langle 2\rangle, \\ \langle 3\rangle\times\langle 1\rangle \end{Bmatrix}$$

# 29. Translation into the aggregated space

• **Theorem** If in an approximation space $(\mathbf{D}, \mathcal{D})$ the atomic elements of $\mathcal{D}$ are two by two disjoint then, for all $r \subseteq \mathbf{D}$ and $a \in \mathcal{D}$,

$$\mathit{apx}\,(r \cap a) = \mathit{apx}\,((\cup_{x \in r} \mathit{apx}\,(\{x\})) \cap a)$$

• **Corollary** If the atomic elements of $\mathcal{D}$ are two by two disjoint, then the mapping $\mathit{agr}_{\mathcal{D}}$ is a translation of the total approximation space $(\mathbf{D}, \mathcal{D})$ into its aggregate $(\mathbf{D}', \mathcal{D}')$.

• **Recalled definition** A *translation* of a total approximation space $(\mathbf{D}, \mathcal{D})$ into an approximation space $(\mathbf{D}', \mathcal{D}')$ is a mapping $\varphi$, of type $\mathcal{P}(\mathbf{D}) \to \mathcal{P}(\mathbf{D}')$, such that,

1. $\varphi(\mathit{apx}_{\mathcal{D}}(r \cap a)) = \mathit{apx}_{\mathcal{D}'}(\varphi(r) \cap \varphi(a))$, for all $r \subseteq \mathbf{D}$ and $a \in \mathcal{D}$,
2. the restriction of $\varphi$ to $\mathcal{D}$ defines an injective mapping of type $\mathcal{D} \to \mathcal{D}'$.

# 30. Moving to machine reals

• Let $(\mathbf{R}, \mathcal{R})$ be the approximation space, where $\mathcal{R}$ is the set of convex subsets of **R** whoses lower and upper bounds, if they exist, belong to

$$\mathbf{F} = \{f_{-k}, f_{-k+1}, \ldots, f_1, f_0, f_1, \ldots, f_{k-1}, f_k\},$$

with $f_i < f_{i+1}$ and $f_{-i} = -f_i$

• The aggregated space of $(\mathbf{R}, \mathcal{R})$ is $(\mathbf{R}', \mathcal{R}')$, where

$$\mathbf{R}' := \left\{ \begin{matrix} \langle -2n-1 \rangle, \\ \langle -2n \rangle, \\ \ldots \\ \langle -1 \rangle, \\ \langle 0 \rangle, \\ \langle 1 \rangle, \\ \ldots \\ \langle 2n \rangle, \\ \langle 2n+1 \rangle \end{matrix} \right\} := \left\{ \begin{matrix} (-\infty, f_{-n}), \\ [f_{-n}, f_{-n}], \\ \ldots \\ (f_{-1}, f_0), \\ [f_0, f_0], \\ (f_0, f_1), \\ \ldots \\ [f_n, f_n], \\ (f_n, +\infty) \end{matrix} \right\}$$

$\mathcal{R}'$ is the set of intervals of $(\mathbf{D}', \preceq)$, with $\langle i \rangle \preceq \langle j \rangle \ \leftrightarrow \ i \leq j$.

• **Remark** If the $f_i's$ are IEEE floating point numbers then $f_i$ is coded as $i$.

# 31. Moving to machine reals, next

- By letting $\phi(a) := \mathit{agr}_{\mathcal{D}'}(a)$, for all convex subset $a$ of **R**, with eventual lower and upper bounds in **F**,

$$
\boxed{\begin{array}{lcl}
\varphi(\emptyset) & = & \emptyset, \\
\varphi([f_i, f_j]) & = & [\langle 2i\rangle, \langle 2j\rangle], \\
\varphi([f_i, f_j)) & = & [\langle 2i\rangle, \langle 2j-1\rangle], \\
\varphi((f_i, f_j]) & = & [\langle 2i+1\rangle, \langle 2j\rangle], \\
\varphi((f_i, f_j)) & = & [\langle 2i+1\rangle, \langle 2j-1\rangle], \\
\varphi([f_i, +\infty)) & = & [\langle 2i\rangle, \langle 2n+1\rangle], \\
\varphi((-\infty, f_j]) & = & [\langle -2n-1\rangle, \langle 2j\rangle], \\
\varphi((f_i, +\infty)) & = & [\langle 2i+1\rangle, \langle 2n+1\rangle], \\
\varphi((-\infty, f_j)) & = & [\langle -2n-1\rangle, \langle 2j-1\rangle], \\
\varphi((-\infty, +\infty)) & = & [\langle -2n-1\rangle, \langle 2n+1\rangle].
\end{array}}
$$

- For the relations

$$
\begin{array}{lcl}
\mathit{times} & := & \{(x, y, z) \in \mathbf{R}^3 \mid z = xy\}, \\
\mathit{ndivision} & := & \{(x, y, z) \in \mathbf{R}^3 \mid y = xz \text{ and } y < 0\}, \\
\mathit{zdivision} & := & \{(x, y, z) \in \mathbf{R}^3 \mid y = xz \text{ and } y = 0\}, \\
\mathit{pdivision} & := & \{(x, y, z) \in \mathbf{R}^3 \mid y = xz \text{ and } y > 0\}
\end{array}
$$

we have

$$\begin{array}{|lcl|}\hline
\mathit{agr}_{\mathcal{R}'(3)}(\mathit{times}) & := & \{(x,y,z) \in (\mathbf{R}')^3 \mid z \in [x\lfloor\times\rfloor y,\ x\lceil\times\rceil y]\}, \\
\mathit{agr}_{\mathcal{R}'(3)}(\mathit{pdivision}) & := & \{(x,y,z) \in (\mathbf{R}')^3 \mid z \in [x\lfloor/\rfloor y,\ x\lceil/\rceil y] \text{ and } y \prec \langle 0\rangle\}, \\
\mathit{agr}_{\mathcal{R}'(3)}(\mathit{zdivision}) & := & (\{\langle 0\rangle\}\times\{\langle 0\rangle\}\times\mathbf{R}') \ \cup\ (\mathbf{R}'\times\{\langle 0\rangle\}\times\{\langle 0\rangle\}), \\
\mathit{agr}_{\mathcal{R}'(3)}(\mathit{ndivision}) & := & \{(x,y,z) \in (\mathbf{R}')^3 \mid z \in [x\lfloor/\rfloor y,\ x\lceil/\rceil y] \text{ and } y \succ \langle 0\rangle\} \\
\hline\end{array}$$

# 32. Discrete multiplication

- Values of $\langle i\rangle \lfloor\times\rfloor \langle j\rangle$

$$\langle i\rangle \lfloor\times\rfloor \langle j\rangle \;:=\; \begin{cases} \text{first value whose condition} & \\ \text{part is satisfied:} & \\ \langle 0\rangle, & i=0 \text{ or } j=0, \\ \langle -i\rangle \lfloor\times\rfloor \langle -j\rangle, & i<0 \text{ and } j<0, \\ -(\langle -i\rangle \lceil\times\rceil \langle j\rangle), & i<0 \text{ and } j>0, \\ -(\langle i\rangle \lceil\times\rceil \langle -j\rangle), & i>0 \text{ and } j<0, \\ \langle 1\rangle, & i=1 \text{ or } j=1, \\ \langle 2n{+}1\rangle, & i=2n{+}1 \text{ ou } j=2n{+}1, \\ \langle 2\alpha(f_{\frac{i}{2}} f_{\frac{j}{2}})\rangle, & i \text{ even, } j \text{ even and } f_{\frac{i}{2}} f_{\frac{j}{2}} \in \mathbf{F}, \\ \langle 2\alpha(f_{\lfloor\frac{i}{2}\rfloor} f_{\lfloor\frac{j}{2}\rfloor}){+}1\rangle, & \text{in the other cases.} \end{cases}$$

- with

$$\begin{array}{lcl} -\langle k\rangle & := & \langle -k\rangle, \\ \alpha(x) & := & \max\{k \in 0..n \mid f_k \le x\}, \\ \beta(x) & := & \max\{k \in 0..n \mid f_k < x\}. \end{array}$$

# 33. Discrete multiplication, next

- Values of $\langle i\rangle \lceil\times\rceil \langle j\rangle$

$$\langle i\rangle \lceil\times\rceil \langle j\rangle \;:=\; \begin{cases} \text{first value whose condition} \\ \text{part is satisfied:} \\ \langle 0\rangle, & i = 0 \text{ or } j = 0, \\ \langle -i\rangle \lceil\times\rceil \langle -j\rangle, & i < 0 \text{ and } j < 0, \\ -(\langle -i\rangle \lfloor\times\rfloor \langle j\rangle), & i < 0 \text{ and } j > 0, \\ -(\langle i\rangle \lfloor\times\rfloor \langle -j\rangle), & i > 0 \text{ and } j < 0, \\ \langle 2n+1\rangle, & i = 2n+1 \text{ ou } j = 2n+1, \\ \langle 1\rangle, & i = 1 \text{ or } j = 1, \\ \langle 2\beta(f_{\frac{i}{2}} f_{\frac{j}{2}})+2\rangle, & i \text{ even, } j \text{ even and } f_{\frac{i}{2}} f_{\frac{j}{2}} \in \mathbf{F}, \\ \langle 2\beta(f_{\lceil\frac{i}{2}\rceil} f_{\lceil\frac{j}{2}\rceil})+1\rangle, & \text{in the other cases.} \end{cases}$$

- with

$$\begin{aligned} -\langle k\rangle &:= \langle -k\rangle, \\ \alpha(x) &:= \max\{k \in 0..n \mid f_k \le x\}, \\ \beta(x) &:= \max\{k \in 0..n \mid f_k < x\}. \end{aligned}$$

# 34. Discrete division

- Values of $\langle i\rangle \lfloor/\rfloor \langle j\rangle$

$$\langle i\rangle \lfloor/\rfloor \langle j\rangle \;:=\; \begin{cases} \text{first value whose condition} \\ \text{part is satisfied:} \\ \langle 0\rangle, & i = 0\,, \\ \langle -i\rangle \lfloor/\rfloor \langle -j\rangle, & i < 0 \text{ and } j < 0, \\ -(\langle -i\rangle \lceil/\rceil \langle j\rangle), & i < 0 \text{ and } j > 0, \\ -(\langle i\rangle \lceil/\rceil \langle -j\rangle), & i > 0 \text{ and } j < 0, \\ \langle 1\rangle, & i = 1 \text{ or } j = 2n{+}1, \\ \langle 2n{+}1\rangle, & i = 2n{+}1 \text{ ou } j = 2n{+}1, \\ \langle 2\alpha(f_{\frac{i}{2}}/f_{\frac{j}{2}})\rangle, & i \text{ even, } j \text{ even and } f_{\frac{i}{2}}/f_{\frac{j}{2}} \in \mathbf{F}, \\ \langle 2\alpha(f_{\lfloor\frac{i}{2}\rfloor}/f_{\lceil\frac{j}{2}\rceil}){+}1\rangle, & \text{in the other cases.} \end{cases}$$

- with

$$\begin{aligned} -\langle k\rangle &:= \langle -k\rangle, \\ \alpha(x) &:= \max\{k \in 0..n \mid f_k \le x\}, \\ \beta(x) &:= \max\{k \in 0..n \mid f_k < x\}. \end{aligned}$$

# 35. Discrete division, next

- Values of $\langle i \rangle \lceil/\rceil \langle j \rangle$

$$\langle i \rangle \lceil/\rceil \langle j \rangle \;:=\; \begin{cases} \text{first value whose condition} \\ \text{part is satisfied:} \\ \langle 0 \rangle, & i = 0\ , \\ \langle -i \rangle \lceil/\rceil \langle -j \rangle, & i < 0 \text{ and } j < 0, \\ -(\langle -i \rangle \lfloor/\rfloor \langle j \rangle), & i < 0 \text{ and } j > 0, \\ -(\langle i \rangle \lfloor/\rfloor \langle -j \rangle), & i > 0 \text{ and } j < 0, \\ \langle 2n+1 \rangle, & i = 2n+1 \text{ or } j = 2n+1, \\ \langle 1 \rangle, & i = 1 \text{ or } j = 2n+1, \\ \langle 2\beta(f_{\frac{i}{2}}/f_{\frac{j}{2}})+2 \rangle, & i \text{ even, } j \text{ even and } f_{\frac{i}{2}}/f_{\frac{j}{2}} \in \mathbf{F}, \\ \langle 2\beta(f_{\lceil\frac{i}{2}\rceil}/f_{\lfloor\frac{j}{2}\rfloor})+1 \rangle, & \text{in the other cases.} \end{cases}$$

- with

$$\begin{aligned} -\langle k \rangle &:= \langle -k \rangle, \\ \alpha(x) &:= \max\{k \in 0..n \mid f_k \le x\}, \\ \beta(x) &:= \max\{k \in 0..n \mid f_k < x\}. \end{aligned}$$

# 36. Solving the multiplication in the machine convex subsets

$$
\textit{\textbf{apx}}_{\mathcal{R}^{(3)}}(\textit{\textbf{times}} \cap a) \; = \;
\left(\begin{array}{l}
\varphi^{-1}\left(\textit{\textbf{apx}}_{\mathcal{R}'}\left(\begin{array}{l} a_1 \cap \textit{\textbf{divz}}\,(a_3, a_2) \cup \\ a_1 \cap \textit{\textbf{divn}}\,(a_3, a_2) \cup \\ a_1 \cap \textit{\textbf{divp}}\,(a_3, a_2) \end{array}\right)\right) \\
\times \\
\varphi^{-1}\left(\textit{\textbf{apx}}_{\mathcal{R}'}\left(\begin{array}{l} a_2 \cap \textit{\textbf{divz}}\,(a_3, a_1) \cup \\ a_2 \cap \textit{\textbf{divn}}\,(a_3, a_1) \cup \\ a_2 \cap \textit{\textbf{divp}}\,(a_3, a_2) \end{array}\right)\right) \\
\times \\
\varphi^{-1}(a_3 \cap \textit{\textbf{mult}}\,(a_1, a_2))
\end{array}\right),
$$

$$
\begin{array}{lcl}
a_i & := & \varphi(\pi_i(a)), \\
\textit{\textbf{divz}}\,(u, v) & := & \text{if } \langle 0 \rangle \in u \text{ and } \langle 0 \rangle \in v \text{ then } [\langle -n \rangle, \langle n \rangle] \text{ else } \emptyset, \\
\textit{\textbf{divn}}\,(u, v) & := & \left[\begin{array}{l} \inf\{x \lfloor / \rfloor y \mid (x, y) \in \textit{\textbf{ends}}\,(u \times (v \cap [\langle -2n-1 \rangle, \langle -1 \rangle]))\}, \\ \sup\{x \lceil / \rceil y \mid (x, y) \in \textit{\textbf{ends}}\,(u \times (v \cap [\langle -2n-1 \rangle, \langle -1 \rangle]))\} \end{array}\right], \\
\textit{\textbf{divp}}\,(u, v) & := & \left[\begin{array}{l} \inf\{x \lfloor / \rfloor y \mid (x, y) \in \textit{\textbf{ends}}\,(u \times (v \cap [\langle 1 \rangle, \langle 2n+1 \rangle]))\}, \\ \sup\{x \lceil / \rceil y \mid (x, y) \in \textit{\textbf{ends}}\,(u \times (v \cap [\langle 1 \rangle, \langle 2n+1 \rangle]))\} \end{array}\right], \\
\textit{\textbf{mult}}\,(u, v) & := & \left[\begin{array}{l} \inf\{x \lfloor \times \rfloor y \mid (x, y) \in \textit{\textbf{ends}}\,(u \times v)\}, \\ \sup\{x \lceil \times \rceil y \mid (x, y) \in \textit{\textbf{ends}}\,(u \times v)\} \end{array}\right].
\end{array}
$$

# 37. Example of discrete multiplication

- $$\mathbf{R}' := \left\{ \begin{array}{c} \langle -3 \rangle, \\ \langle -2 \rangle, \\ \langle -1 \rangle, \\ \langle 0 \rangle, \\ \langle 1 \rangle, \\ \langle 2 \rangle, \\ \langle 3 \rangle \end{array} \right\} := \left\{ \begin{array}{c} (-\infty, -1), \\ [-1, -1], \\ (-1, 0), \\ [0, 0], \\ (0, 1), \\ [1, 1], \\ (1, +\infty) \end{array} \right\} \qquad \mathbf{F} := \left\{ \begin{array}{c} \\ -1, \\ \\ 0, \\ \\ 1 \\ \\ \end{array} \right\}$$

- **Table of $\langle i \rangle \lfloor\times\rfloor \langle j \rangle$**

| | $\langle -3 \rangle$ | $\langle -2 \rangle$ | $\langle -1 \rangle$ | $\langle 0 \rangle$ | $\langle 1 \rangle$ | $\langle 2 \rangle$ | $\langle 3 \rangle$ |
|---|---|---|---|---|---|---|---|
| $\langle -3 \rangle$ | $\langle 3 \rangle$ | $\langle 3 \rangle$ | $\langle 1 \rangle$ | $\langle 0 \rangle$ | $\langle -3 \rangle$ | $\langle -3 \rangle$ | $\langle -3 \rangle$ |
| $\langle -2 \rangle$ | $\langle 3 \rangle$ | $\langle 2 \rangle$ | $\langle 1 \rangle$ | $\langle 0 \rangle$ | $\langle -1 \rangle$ | $\langle -2 \rangle$ | $\langle -3 \rangle$ |
| $\langle -1 \rangle$ | $\langle 1 \rangle$ | $\langle 1 \rangle$ | $\langle 1 \rangle$ | $\langle 0 \rangle$ | $\langle -1 \rangle$ | $\langle -1 \rangle$ | $\langle -3 \rangle$ |
| $\langle 0 \rangle$ | $\langle 0 \rangle$ | $\langle 0 \rangle$ | $\langle 0 \rangle$ | $\langle 0 \rangle$ | $\langle 0 \rangle$ | $\langle 0 \rangle$ | $\langle 0 \rangle$ |
| $\langle 1 \rangle$ | $\langle -3 \rangle$ | $\langle -1 \rangle$ | $\langle -1 \rangle$ | $\langle 0 \rangle$ | $\langle 1 \rangle$ | $\langle 1 \rangle$ | $\langle 1 \rangle$ |
| $\langle 2 \rangle$ | $\langle -3 \rangle$ | $\langle -2 \rangle$ | $\langle -1 \rangle$ | $\langle 0 \rangle$ | $\langle 1 \rangle$ | $\langle 2 \rangle$ | $\langle 3 \rangle$ |
| $\langle 3 \rangle$ | $\langle -3 \rangle$ | $\langle -3 \rangle$ | $\langle -3 \rangle$ | $\langle 0 \rangle$ | $\langle 1 \rangle$ | $\langle 3 \rangle$ | $\langle 3 \rangle$ |

- **Table of $\langle i \rangle \lceil\times\rceil \langle j \rangle$**

|  | $\langle -3\rangle$ | $\langle -2\rangle$ | $\langle -1\rangle$ | $\langle 0\rangle$ | $\langle 1\rangle$ | $\langle 2\rangle$ | $\langle 3\rangle$ |
|---|---|---|---|---|---|---|---|
| $\langle -3\rangle$ | $\langle 3\rangle$ | $\langle 3\rangle$ | $\langle 3\rangle$ | $\langle 0\rangle$ | $\langle -1\rangle$ | $\langle -3\rangle$ | $\langle -3\rangle$ |
| $\langle -2\rangle$ | $\langle 3\rangle$ | $\langle 2\rangle$ | $\langle 1\rangle$ | $\langle 0\rangle$ | $\langle -1\rangle$ | $\langle -2\rangle$ | $\langle -3\rangle$ |
| $\langle -1\rangle$ | $\langle 3\rangle$ | $\langle 1\rangle$ | $\langle 1\rangle$ | $\langle 0\rangle$ | $\langle -1\rangle$ | $\langle -1\rangle$ | $\langle -1\rangle$ |
| $\langle 0\rangle$ | $\langle 0\rangle$ | $\langle 0\rangle$ | $\langle 0\rangle$ | $\langle 0\rangle$ | $\langle 0\rangle$ | $\langle 0\rangle$ | $\langle 0\rangle$ |
| $\langle 1\rangle$ | $\langle -1\rangle$ | $\langle -1\rangle$ | $\langle -1\rangle$ | $\langle 0\rangle$ | $\langle 1\rangle$ | $\langle 1\rangle$ | $\langle 3\rangle$ |
| $\langle 2\rangle$ | $\langle -3\rangle$ | $\langle -2\rangle$ | $\langle -1\rangle$ | $\langle 0\rangle$ | $\langle 1\rangle$ | $\langle 2\rangle$ | $\langle 3\rangle$ |
| $\langle 3\rangle$ | $\langle -3\rangle$ | $\langle -3\rangle$ | $\langle -1\rangle$ | $\langle 0\rangle$ | $\langle 3\rangle$ | $\langle 3\rangle$ | $\langle 3\rangle$ |

# 38. Example of discrete division

- $$\mathbf{R}' := \left\{\begin{matrix}\langle -3\rangle, \\ \langle -2\rangle, \\ \langle -1\rangle, \\ \langle 0\rangle, \\ \langle 1\rangle, \\ \langle 2\rangle, \\ \langle 3\rangle\end{matrix}\right\} := \left\{\begin{matrix}(-\infty,-1), \\ [-1,-1], \\ (-1,0), \\ [0,0], \\ (0,1), \\ [1,1], \\ (1,+\infty)\end{matrix}\right\} \qquad \mathbf{F} := \left\{\begin{matrix}-1, \\ \\ 0, \\ \\ 1\end{matrix}\right\}$$

- **Table of $\langle i\rangle \lfloor/\rfloor \langle j\rangle$**

| | $\langle -3\rangle$ | $\langle -2\rangle$ | $\langle -1\rangle$ | $\langle 1\rangle$ | $\langle 2\rangle$ | $\langle 3\rangle$ |
|---|---|---|---|---|---|---|
| $\langle -3\rangle$ | $\langle 1\rangle$ | $\langle 3\rangle$ | $\langle 3\rangle$ | $\langle -3\rangle$ | $\langle -3\rangle$ | $\langle -3\rangle$ |
| $\langle -2\rangle$ | $\langle 1\rangle$ | $\langle 2\rangle$ | $\langle 3\rangle$ | $\langle -3\rangle$ | $\langle -2\rangle$ | $\langle -1\rangle$ |
| $\langle -1\rangle$ | $\langle 1\rangle$ | $\langle 1\rangle$ | $\langle 1\rangle$ | $\langle -3\rangle$ | $\langle -1\rangle$ | $\langle -1\rangle$ |
| $\langle 0\rangle$ | $\langle 0\rangle$ | $\langle 0\rangle$ | $\langle 0\rangle$ | $\langle 0\rangle$ | $\langle 0\rangle$ | $\langle 0\rangle$ |
| $\langle 1\rangle$ | $\langle -1\rangle$ | $\langle -1\rangle$ | $\langle -3\rangle$ | $\langle 1\rangle$ | $\langle 1\rangle$ | $\langle 1\rangle$ |
| $\langle 2\rangle$ | $\langle -1\rangle$ | $\langle -2\rangle$ | $\langle -3\rangle$ | $\langle 3\rangle$ | $\langle 2\rangle$ | $\langle 1\rangle$ |
| $\langle 3\rangle$ | $\langle -3\rangle$ | $\langle -3\rangle$ | $\langle -3\rangle$ | $\langle 3\rangle$ | $\langle 3\rangle$ | $\langle 1\rangle$ |

• **Table of $\langle i \rangle \lceil / \rceil \langle j \rangle$**

| | $\langle -3 \rangle$ | $\langle -2 \rangle$ | $\langle -1 \rangle$ | $\langle 1 \rangle$ | $\langle 2 \rangle$ | $\langle 3 \rangle$ |
|---|---|---|---|---|---|---|
| $\langle -3 \rangle$ | $\langle 3 \rangle$ | $\langle 3 \rangle$ | $\langle 3 \rangle$ | $\langle -3 \rangle$ | $\langle -3 \rangle$ | $\langle -1 \rangle$ |
| $\langle -2 \rangle$ | $\langle 1 \rangle$ | $\langle 2 \rangle$ | $\langle 3 \rangle$ | $\langle -3 \rangle$ | $\langle -2 \rangle$ | $\langle -1 \rangle$ |
| $\langle -1 \rangle$ | $\langle 1 \rangle$ | $\langle 1 \rangle$ | $\langle 3 \rangle$ | $\langle -1 \rangle$ | $\langle -1 \rangle$ | $\langle -1 \rangle$ |
| $\langle 0 \rangle$ | $\langle 0 \rangle$ | $\langle 0 \rangle$ | $\langle 0 \rangle$ | $\langle 0 \rangle$ | $\langle 0 \rangle$ | $\langle 0 \rangle$ |
| $\langle 1 \rangle$ | $\langle -1 \rangle$ | $\langle -1 \rangle$ | $\langle -1 \rangle$ | $\langle 3 \rangle$ | $\langle 1 \rangle$ | $\langle 1 \rangle$ |
| $\langle 2 \rangle$ | $\langle -1 \rangle$ | $\langle -2 \rangle$ | $\langle -3 \rangle$ | $\langle 3 \rangle$ | $\langle 2 \rangle$ | $\langle 1 \rangle$ |
| $\langle 3 \rangle$ | $\langle -1 \rangle$ | $\langle -3 \rangle$ | $\langle -3 \rangle$ | $\langle 3 \rangle$ | $\langle 3 \rangle$ | $\langle 3 \rangle$ |

# Extension of an approximation space

# 39. Adherence

• **Definition**. Let $(\mathbf{D}, \mathcal{D})$ be an approximation space and $r$ a subset of **D**. The *adherence* of $r$ is denoted and defined by

$$\mathit{adh}_{\mathcal{D}}(r) := \{x \in \mathbf{D} \,|\, r \cap a \neq \emptyset, \text{ for all } a \in \mathcal{D} \text{ with } x \in a\}$$

• **Properties** For all subsets $r, s$ of **D** and all element $a$ of $\mathcal{D}$:

$$\begin{array}{lll} \text{(i)} & r \cap a = \emptyset \;\; \rightarrow \;\; \mathit{adh}\,(r) \cap a = \emptyset, & \\ \text{(ii)} & r \subseteq a = \emptyset \;\; \rightarrow \;\; \mathit{adh}\,(r) \subseteq a, & \text{if 2 holds,} \\ \text{(iii)} & \mathit{adh}\,(r \cup s) = \mathit{adh}\,(r) \cup \mathit{adh}\,(s), & \text{if 1 holds,} \\ \text{(iv)} & \mathit{apx}\,(r \cap a) = \mathit{apx}\,(\mathit{adh}\,(r) \cap a), & \text{if 1 and 2 hold.} \end{array}$$

where 1 and 2 are the conditions:

1. the set $\mathcal{D}$ is closed for finite intersection,
2. any subset of $\mathbf{D}'$ of the form $\mathbf{D}' - a'$, with $a' \in \mathcal{D}'$, can be written as a union of elements of $\mathcal{D}'$,

# 40. Translation into an extended space

• **Theorem** Let $(\mathbf{D}, \mathcal{D})$, $(\mathbf{D}', \mathcal{D}')$ be two approximation, spaces, the first one being total, such that

1. the set $\mathcal{D}'$ is closed for finite intersection,
2. any subset of $\mathbf{D}'$ of the form $\mathbf{D}' - a'$, with $a' \in \mathcal{D}'$, can be written as a union of elements of $\mathcal{D}'$,
3. the restriction of $\mathit{adh}_{\mathcal{D}'}$ to $\mathcal{D}$ defines a bijection of type $\mathcal{D} \rightarrow \mathcal{D}'$,
4. $a = \mathit{adh}_{\mathcal{D}'}(a) \cap \mathbf{D}$, for all $a \in \mathcal{D}$.

The mapping $r \mapsto \mathit{adh}_{\mathcal{D}'}(r)$, of type $\mathcal{P}(\mathbf{D}) \rightarrow \mathcal{P}(\mathbf{D}')$, is then a translation of the first approximation space into the second.

• **Recalled definition** A *translation* of a total approximation space $(\mathbf{D}, \mathcal{D})$ into an approximation space $(\mathbf{D}', \mathcal{D}')$ is a mapping $\varphi$, of type $\mathcal{P}(\mathbf{D}) \rightarrow \mathcal{P}(\mathbf{D}')$, such that,

1. $\varphi(\mathit{apx}_{\mathcal{D}}(r \cap a)) = \mathit{apx}_{\mathcal{D}'}(\varphi(r) \cap \varphi(a))$, for all $r \subseteq \mathbf{D}$ and $a \in \mathcal{D}$,
2. the restriction of $\varphi$ to $\mathcal{D}$ defines an injective mapping of type $\mathcal{D} \rightarrow \mathcal{D}'$.

# 41. Extension of R

- We extend the approximation space

$$\begin{pmatrix}\mathbf{R}, \\ \mathcal{R}\end{pmatrix} := \begin{pmatrix}\text{set of reals,} \\ \text{set of convex subsets of } \mathbf{R}\end{pmatrix}$$

- into the approximation space

$$\begin{pmatrix}\mathbf{R}', \\ \mathcal{R}'\end{pmatrix} := \begin{pmatrix}\mathbf{R} \cup \mathbf{R}^- \cup \mathbf{R}^+ \cup \{-\infty, +\infty\}, \\ \{[x,y] \mid \; x \in \{-\infty\} \cup \mathbf{R} \cup \mathbf{R}^+ \text{ and } y \in \{+\infty\} \cup \mathbf{R} \cup \mathbf{R}^-\}\end{pmatrix}$$

- with

$$\begin{aligned} \mathbf{R}^- &:= \{x^- \mid x \in \mathbf{R}\}, \\ \mathbf{R}^+ &:= \{x^+ \mid x \in \mathbf{R}\} \end{aligned}$$

and such that, for all elements $x, y$ of **R,**

$$x < y \;\; \leftarrow \;\; -\infty < x^- < x < x^+ < y^- < y < y^+ < +\infty$$

# 42. Extension of R, next

• For all convex subset $a$ of $\mathbf{R}$, the set $\varphi(a) := \mathit{adh}_{\mathcal{D}'}(a)$ is the following interval of $\mathbf{R}'$:

$$\boxed{\begin{array}{lcl}
\varphi(\emptyset) & = & \emptyset, \\
\varphi([e,d]) & = & [e,d], \\
\varphi([e,d)) & = & [e,d^-], \\
\varphi((e,d]) & = & [e^+,d], \\
\varphi((e,d)) & = & [e^+,d^-], \\
\varphi([e,+\infty)) & = & [e,+\infty], \\
\varphi((-\infty,d]) & = & [-\infty,d], \\
\varphi((e,+\infty)) & = & [e^+,+\infty], \\
\varphi((-\infty,d)) & = & [-\infty,d^-], \\
\varphi((-\infty,+\infty)) & = & [-\infty,+\infty].
\end{array}}$$

• For the relations

$$\begin{array}{lcl}
\mathit{times} & := & \{(x,y,z) \in \mathbf{R}^3 \mid z = xy\}, \\
\mathit{ndivision} & := & \{(x,y,z) \in \mathbf{R}^3 \mid y = xz \text{ and } y < 0\}, \\
\mathit{zdivision} & := & \{(x,y,z) \in \mathbf{R}^3 \mid y = xz \text{ and } y = 0\}, \\
\mathit{pdivision} & := & \{(x,y,z) \in \mathbf{R}^3 \mid y = xz \text{ and } y > 0\}.
\end{array}$$

- we have

$$\begin{aligned}
adh_{\mathcal{R}'(3)}(\mathit{times}) &:= \{(x,y,z) \in (\mathbf{R}')^3 \mid z \in [x \lfloor\times\rfloor y,\ x \lceil\times\rceil y]\},\\
adh_{\mathcal{R}'(3)}(\mathit{pdivision}) &:= \{(x,y,z) \in (\mathbf{R}')^3 \mid z \in [x \lfloor/\rfloor y,\ x \lceil/\rceil y] \text{ and } y < 0\},\\
adh_{\mathcal{R}'(3)}(\mathit{zdivision}) &:= (\{0\}\times\{0\}\times\mathbf{R}) \ \cup\ (\{\mathbf{R}\}\times\{0\}\times\{0\}),\\
adh_{\mathcal{R}'(3)}(\mathit{ndivision}) &:= \{(x,y,z) \in (\mathbf{R}')^3 \mid z \in [x \lfloor/\rfloor y,\ x \lceil/\rceil y] \text{ and } y > 0\}.
\end{aligned}$$

# 43. Extended multiplication and division

- $\boldsymbol{x \lfloor\times\rfloor y}$

$$x \lfloor\times\rfloor y \;:=\; \begin{cases} \text{first value whose condition} \\ \text{part is satisfied:} \\ (-x) \lfloor\times\rfloor (-y), & x < 0 \text{ and } y < 0, \\ -((-x) \lceil\times\rceil y), & x < 0 \text{ and } y > 0, \\ -(x \lceil\times\rceil (-y)), & x > 0 \text{ and } y < 0, \\ 0, & x = 0 \text{ or } y = 0, \\ 0^+, & x = 0^+ \text{ or } y = 0^+, \\ +\infty, & x = +\infty \text{ or } y = +\infty, \\ (\mathit{rp}\,(x)\mathit{rp}\,(y))^-, & x \in \mathbf{R}^- \text{ or } y \in \mathbf{R}^-, \\ (\mathit{rp}\,(x)\mathit{rp}\,(y))^+, & x \in \mathbf{R}^+ \text{ or } y \in \mathbf{R}^+, \\ xy, & x \in \mathbf{R} \text{ and } y \in \mathbf{R}. \end{cases}$$

- with

$$-x \;:=\; \begin{cases} +\infty, & \text{if } x = -\infty, \\ (-\mathit{rp}\,(x))^-, & \text{if } x \in \mathbf{R}^+, \\ -x, & \text{if } x \in \mathbf{R}, \\ (-\mathit{rp}\,(x))^+, & \text{if } x \in \mathbf{R}^-, \\ -\infty, & \text{if } x = +\infty. \end{cases}$$

- and, for all $x \in \mathbf{R}$,

$$\boxed{\begin{array}{lcl} \mathit{rp}\,(x^-) & := & x, \\ \mathit{rp}\,(x) & := & x, \\ \mathit{rp}\,(x^+) & := & x. \end{array}}$$

# 44. Extended multiplication and division, next

- $\boldsymbol{x \lceil\times\rceil y}$

$$x \lceil\times\rceil y := \begin{cases} \text{first value whose condition} \\ \text{part is satisfied:} \\ (-x)\lceil\times\rceil(-y), & x < 0 \text{ and } y < 0, \\ -((-x)\lfloor\times\rfloor y), & x < 0 \text{ and } y > 0, \\ -(x\lfloor\times\rfloor(-y)), & x > 0 \text{ and } y < 0, \\ 0, & x = 0 \text{ ou } y = 0, \\ +\infty, & x = +\infty \text{ ou } y = +\infty, \\ 0^+, & x = 0^+ \text{ ou } y = 0^+, \\ (\mathit{rp}\,(x)\mathit{rp}\,(y))^+, & x \in \mathbf{R}^+ \text{ ou } y \in \mathbf{R}^+, \\ (\mathit{rp}\,(x)\mathit{rp}\,(y))^-, & x \in \mathbf{R}^- \text{ ou } y \in \mathbf{R}^-, \\ xy, & x \in \mathbf{R} \text{ and } y \in \mathbf{R}. \end{cases}$$

- $\boldsymbol{x} \lceil / \rceil \boldsymbol{y}$ **and** $\boldsymbol{x} \lfloor / \rfloor \boldsymbol{y}$

$$
\begin{aligned}
x \lfloor / \rfloor y &:= x \lfloor \times \rfloor (1/y), \\
x \lceil / \rceil y &:= x \lceil \times \rceil (1/y), \\
1/x &:= \begin{cases} -\infty, & \text{if } x = 0^-, \\ 0^-, & \text{if } x = -\infty, \\ (\boldsymbol{rp}\,(1/x))^-, & \text{if } x \in \mathbf{R}^+, \\ 1/x, & \text{if } x \in \mathbf{R}, \\ (\boldsymbol{rp}\,(1/x))^+, & \text{if } x \in \mathbf{R}^-, \\ 0^+, & \text{if } x = +\infty, \\ +\infty, & \text{if } x = 0^+. \end{cases}
\end{aligned}
$$

# 45. Solving the multiplication in the convex subsets

We conclude that, in the approximation space $(\mathbf{R}, \mathcal{R})$ of the reals by convex subsets, if $a$ is a $3$-box,

$$\mathbf{\mathit{apx}}_{\mathcal{R}^{(3)}}(\mathbf{\mathit{times}} \cap a) \;=$$

$$\begin{pmatrix} \varphi^{-1}\left(\mathbf{\mathit{apx}}_{\mathcal{R}'}\begin{pmatrix} a_1 \cap \mathbf{\mathit{divz}}(a_3, a_2) \cup \\ a_1 \cap \mathbf{\mathit{divn}}(a_3, a_2) \cup \\ a_1 \cap \mathbf{\mathit{divp}}(a_3, a_2) \end{pmatrix}\right) \\ \times \\ \varphi^{-1}\left(\mathbf{\mathit{apx}}_{\mathcal{R}'}\begin{pmatrix} a_2 \cap \mathbf{\mathit{divz}}(a_3, a_1) \cup \\ a_2 \cap \mathbf{\mathit{divn}}(a_3, a_1) \cup \\ a_2 \cap \mathbf{\mathit{divp}}(a_3, a_2) \end{pmatrix}\right) \\ \times \\ \varphi^{-1}(a_3 \cap \mathbf{\mathit{mult}}(a_1, a_2)) \end{pmatrix},$$

$$\begin{aligned}
a_i &:= \varphi(\pi_i(a)), \\
\mathbf{\mathit{divz}}(u, v) &:= \text{if } 0 \in u \text{ and } 0 \in v \text{ then } [-\infty, +\infty] \text{ else } \emptyset, \\
\mathbf{\mathit{divn}}(u, v) &:= \begin{bmatrix} \inf\{x \lfloor / \rfloor y \mid (x, y) \in \mathbf{\mathit{ends}}(u \times (v \cap [-\infty, 0^-]))\}, \\ \sup\{x \lceil / \rceil y \mid (x, y) \in \mathbf{\mathit{ends}}(u \times (v \cap [-\infty, 0^-]))\} \end{bmatrix}, \\
\mathbf{\mathit{divp}}(u, v) &:= \begin{bmatrix} \inf\{x \lfloor / \rfloor y \mid (x, y) \in \mathbf{\mathit{ends}}(u \times (v \cap [0^+, +\infty]))\}, \\ \sup\{x \lceil / \rceil y \mid (x, y) \in \mathbf{\mathit{ends}}(u \times (v \cap [0^+, +\infty]))\} \end{bmatrix}, \\
\mathbf{\mathit{mult}}(u, v) &:= \begin{bmatrix} \inf\{x \lfloor \times \rfloor y \mid (x, y) \in \mathbf{\mathit{ends}}(u \times v)\}, \\ \sup\{x \lceil \times \rceil y \mid (x, y) \in \mathbf{\mathit{ends}}(u \times v)\} \end{bmatrix}.
\end{aligned}$$

# Complement

# 46. Related work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# 47. About unions and intersections, eventually infinite

• Let $\mathcal{D}$ be any set of sets. The *union* of the elements of $\mathcal{D}$ is denoted and defined by

$$\cup\mathcal{D} := \{x \,|\, \text{there exists } a \in \mathcal{D}, \text{ with } x \in a\}.$$

• The *intersection* of the elements of $\mathcal{D}$ is denoted and defined by

$$\cap\mathcal{D} \;:=\; \{x \in \cup\mathbf{D} \,|\, \text{for all } a \in \mathcal{D}, \text{ we have } x \in a\}.$$

• **Definition** The set $\mathcal{D}$ is *closed by intersection* if for any non empty subset $\mathcal{D}'$ of $\mathcal{D}$ the set $\cap\mathcal{D}'$ belongs $\mathcal{D}$.

• **Definition** The set $\mathcal{D}$ is *closed by finite intersection* if for all finite non empty subset $\mathcal{D}'$ of $\mathcal{D}$ the set $\cap\mathcal{D}'$ belongs to $\mathcal{D}$.

• **Property** A sufficient condition in order that $\mathcal{D}$ is closed by intersection is that

1. $\mathcal{D}$ is closed by finite intersection and
2. any sequence of elements of $\mathcal{D}$, strictly decreasing for inclusion, is finite.