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Abstract

The goal of this thesis is the study of a harmonious way to combine any first order theory with the theory of finite or infinite trees. For that:

First of all, we introduce two classes of theories that we call *infinite-decomposable* and *zero-infinite-decomposable*. We show that these theories are complete and accept a decision procedure which for every proposition gives either *true* or *false*. We show also that these classes of theories contain a large number of fundamental theories used in computer science, we can cite for example: the theory of additive rational or real numbers, the theory of the linear dense order without endpoints, the theory of finite or infinite trees, the construction of trees on an ordered set, and a combination of trees and ordered additive rational or real numbers.

We give then an automatic way to combine any first order theory T with the theory of finite or infinite trees. A such hybrid theory is called *extension into trees* of the theory T and is denoted by T^*. After having defined the axiomatization of T^* using those of T, we define a new class of theories that we call *flexible* and show that if T is flexible then T^* is zero-infinite-decomposable and thus complete. The flexible theories are first order theories having elegant properties which enable us to handle easily first order formulas. We show among other theories that the theory T_{ad} of ordered additive rational numbers is flexible and thus that the extension into trees T^*_{ad} of T_{ad} is complete.

Finally, we end this thesis by a general algorithm for solving efficiently first order constraints in T^*_{ad}. The algorithm is given in the form of 28 rewriting rules which transform every formula φ, which can possibly contain free variables, into a disjunction ϕ of solved formulas equivalent to φ in T^*_{ad} and such that ϕ is either the formula *true*, or the formula *false*, or a formula having at least one free variable and being equivalent neither to *true* nor to *false* in T^*_{ad}. Moreover, the solutions of the free variables of ϕ are expressed in a clear and explicit way in ϕ.

Keywords: Theory of finite or infinite trees, Complete theory, Combination of theories, Solving first order constraints, Rewriting rules.

Contents

Introduction

The algebra of finite or infinite trees plays a fundamental role in computer science: it is a model for data structures, program schemes and program executions. As early as 1976, G. Huet proposed an algorithm for unifying infinite terms, that is solving equations in that algebra [28]. B. Courcelle has studied the properties of infinite trees in the scope of recursive program schemes [12]. A. Colmerauer has described the execution of Prolog II, III and IV programs in terms of solving equations and disequations in that algebra [5, 6, 1]. The unification of finite terms, i.e. the resolution of conjunctions of equations in the theory of finite trees has first been studied by A. Robinson [38]. Some better algorithms with better complexities have been proposed after by M.S. Paterson and M.N.Wegman [36] and A. Martelli and U. Montanari [35]. The resolution of conjunctions of equations in the theory of infinite trees has been studied by G. Huet [28], by A. Colmerauer [4, 5] and by J. Jaffar [29]. The resolution of conjunctions of equations and disequations in the theory of possibly infinite trees has been studied by A. Colmerauer [5] and H.J. Bürckert [2]. An incremental algorithm for solving conjunctions of equations and disequations on rational trees has been proposed after by V.Ramachandran and P. Van Hentenryck [37]. On the other hand, there exists an algorithm for elimination of quantifications which transforms a first-order formula into a boolean combination of simple constraints. We can refer to the work of M.J. Maher [33] and H. Comon [11].

M.J. Maher has axiomatized all the cases by complete first-order theories with infinite set of function symbols [33]. It is this theory which has been the starting point of our works. After having studied its properties we have created two classes of theories that we call *infinite-decomposable* and *zero-infinite-decomposable* and have shown that a lot of fundamental theories used in computer science belong to these classes. We can cite for example: the theory of finite trees, the theory of infinite trees, the theory of finite or infinite trees [19], the theory of the linear dense order without endpoints, the theory of additive rational or real numbers, the construction of trees on an ordered set [23] and the combination of finite or infinite trees and ordered additive rational or real numbers [24]. The first intuitions behind these classes of theories come from the works of T. Dao [16] which has proposed a general algorithm solving first order constraints in the theory of finite or infinite trees [16],[19] using a basic simplification of quantified conjunctions of atomic formulas. We have then generalized this simplification by showing that in every infinite-decomposable or zero-infinite-decomposable theory, it is always possible to decompose a series of existential quantifications on a conjunction of atomic formulas, into three embedded sequences having elegant properties which can be expressed using four special quantifiers denoted by $\exists?$, $\exists!$, $\exists_{\infty}^{\Psi(u)}$, $\exists_{o\infty}^{\Psi(u)}$ and called *at-most-one*, *one-and-only-one*, *infinite* and *zero-infinite*. While the quantifiers $\exists?$, $\exists!$, are just convenient notations, the quantifiers $\exists_{\infty}^{\Psi(u)}$, $\exists_{o\infty}^{\Psi(u)}$ express a property which can not be expressed in the first order level. The names *infinite-decomposable* and *zero-infinite-decomposable* have not been chosen randomly. In fact, a zero-infinite decomposable theory is decomposed using only the quantifiers $\exists?$, $\exists!$, $\exists_{\infty}^{\Psi(u)}$, while a zero-infinite-decomposable theory is decomposed using the quantifiers $\exists ?$, $\exists !$, $\exists_{o\,\infty}^{\Psi(u)}$. After having studied the properties of these special quantifiers, we have show the completeness of all these classes of theories and given a decision procedure in the form of general rewriting rules which for every proposition give either true or false.

We have then interested ourselves to the problem of combination of theories together with non-disjoint signatures and more exactly to the combination of any first order theory T with the theory of finite or infinite trees. This work reflects essentially to Prolog III and Prolog IV which have been modeled by A. Colmerauer [6] using a combination of trees, rational numbers, booleans and intervals. One of the major difficulties in this combination resides in the fact that the two combined theories can have non-disjoint signatures, i.e. the existence of at least one function or relation symbol having two completely different behaviors whether we handle the theory T or the theory of finite or infinite trees. Moreover, the theory of finite or infinite trees does not accept full elimination of quantifiers which makes the completeness of any combination with it not evident. For that, we would first to define a semantic meaning for this combination and then to give a harmonious axiomatization of the new hybrid theory. In our point of view, a combination of a theory T with the theory of finite or infinite trees is nothing other than an extension into trees of the elements of the models of the theory T. Thus, the axiomatization of the extension into trees of T, denoted by T^*, proceeds essentially from the axiomatization of the theory T and the three axioms of Michael Maher on the theory of finite or infinite trees [33] by introducing typing constraints that distinguish the tree elements from the others. To show the completeness of T^*, we have introduced the class of the flexible theories and shown that if T is flexible then its extension into trees, i.e. T^*, is a zero-infinite-decomposable theory and thus a complete theory. The flexible theories are first order theories having elegant properties which enable us to handle easily first order formulas.

Once these results obtained, we have interested ourselves to build a general algorithm for solving first order hybrid constraints and which gives solutions of the free variables in a clear and explicit way. For us, solving a constraint φ, which can possibly contain free variables, in T^*, means to transform the first order formula φ into a disjunction ϕ of solved formulas, equivalent to φ in T^* and such that ϕ is, either the formula *true*, or the formula *false*, or a formula having at least one free variable and being equivalent neither to *true* nor to *false* in T^*. Of course, the two decision procedures given for the infinite-decomposable and zero-infinite-decomposable theories are not able to solve general first order constraints since they can only decide the validity or not validity of propositions (sentences). They are not able to express the solutions of the solved formula in a clear and explicit way and do not warrant that any disjunction of solved formulas containing at least one free variable is equivalent neither to true nor to false in T^*. We have then chosen the extension into trees T^*_{ad} of the theory T_{ad} of ordered additive rational numbers and have given an efficient algorithm solving any first order constraint in T^*_{ad}. One of the major difficulties in this work resides in the fact that (1) every algorithm solving only propositions in the theory of finite or infinite trees has a non-elementary complexity in the form of tower of powers of 2 [41], (2) the theory of finite or infinite trees does not accept full elimination of quantifiers, (3) the function symbols $+$ and $-$ have two completely different behaviors whether they handle rationals or trees. For example, the individual $+(1,1)$ is the rational 2, while the individual $+(1, f_0)$ is the tree whose root is labeled $+$ and whose suns are 1 and the tree reduced to a leaf labeled by f_0.

This thesis contains five chapters followed by a conclusion. In Chapter 1 we recall the basic notions of first order logic and give a sufficient condition for the completeness of any first order theory.

In Chapter 2, we give a formal definition of the infinite-decomposable theories. The main idea behind this definition consists in decomposing any series of existential quantifications on a conjunction of atomic formulas, into three embedded sequences having elegant properties which can be expressed using the special quantifiers $\exists ?$, $\exists !$, $\exists_{\infty}^{\Psi(u)}$. After having given the properties of these special quantifiers we show the completeness of any infinite-decomposable theory using the sufficient condition of completeness of first order theories given in Chapter 1. We give also a decision procedure in any infinite-decomposable theory T, in the form of five rewriting rules which for every proposition give either *true* or *false* in T. The correctness of our algorithm is another proof of the completeness of the infinite-decomposable theories. We end this chapter by an application to the theory $\mathcal{T}$ of finite or infinite trees. We show that $\mathcal{T}$ is infinite-decomposable and give two examples of solving propositions in $\mathcal{T}$.

In Chapter 3, we present the class of the *zero-infinite-decomposable* theories which is an extension of the infinite-decomposable theories by replacing the infinite quantifier by the zero-infinite quantifier. We show the completeness of any zero-infinite-decomposable theory using the sufficient condition of completeness of first order theories given in Chapter 1. We give also a property which links the infinite-decomposable theories to the zero-infinite-decomposable theories and show that while all the infinite-decomposable theories given in Chapter 2 are also zero-infinite-decomposable, the simple theory of the linear dense order is not infinite-decomposable but zero-infinite-decomposable. We present then a decision procedure for every zero-infinite-decomposable theory T, in the form of six rewriting rules which for every proposition give either *true* or *false* in T. This algorithm contains a new rule comparing with those of the infinite-decompsoable theories due to the zero-infnite quantifier which enables only a partial elimination of quantifiers while the infinite-quantifier enables a full elimination of quantifiers. We end this chapter by an application to the construction of trees on an ordered set. This theory, denoted by $\mathcal{T}_{ord}$, is a complete axiomatization of the construction of trees on a set of individuals together with a linear dense order relation without endpoints. After having presented the axiomatization of $\mathcal{T}_{ord}$, we show its zero-infinite-decomposability and end by an example solving propositions in $\mathcal{T}_{ord}$.

In Chapter 4, we give an automatic way to combine any first order theory T with the theory of finite or infinite trees. The axiomatization of the extension into trees of T, denoted by T^*, is made essentially from the axiomatization of the theory T and the three axioms of Michael Maher of the theory of finite or infinite trees [33] as well as a full system of typing constraints. For each theory T^* we give a formal definition of the standard model M^* of T^* using the standard model M of T. To show the completeness of the theory T^*, we introduce the *flexible* theories and show that if T is flexible, then its extension into trees, i.e. T^*, is zero-infinite-decomposable and thus complete. We end this chapter by an application to the extension into trees T^*_{ad} of the theory T_{ad} of ordered additive rational numbers. We show that T_{ad} is flexible and thus T^*_{ad} is complete.

Finally, in Chapter 5, we give a general algorithm solving any first order constraint in the theory T^*_{ad}. After having defined the meaning of a first order constraint in T^*_{ad}, we present our solver in the form of 28 rewriting rules which transform every formula φ into a disjunction ϕ of solved formulas, equivalent to φ in T^*_{ad} and such that ϕ is, either the formula *true*, or the formula *false*, or a formula having at least one free variable and being equivalent neither to *true* nor to *false* in T^*_{ad}. While the two decision procedures given in Chapter 2 and 3 solve only propositions, this algorithm gives the solutions of the free variables in a clear and explicit way and is able to check if a formula having at least one free variable is always true or false in T^*_{ad}. It also warrants that every disjunction ϕ of solved formulas having at least one free variable accepts two explicit instantiations ϕ_1 and ϕ_2 such that for every model M^*_{ad} of T^*_{ad} we have $M^*_{ad} \models \phi_1$ and

$M^*_{ad} \models \neg\phi_2$. We end this chapter by an example of solving a constraint having two free variables and being always equivalent in *false* in T^*_{ad}.

The sufficient condition for the completeness of first order theories given in Chapter 1, the quantifiers $\exists^{\Psi(u)}_{\infty}$ and $\exists^{\Psi(u)}_{o\,\infty}$, the classes of the infinite-decomposable and zero-infinite-decomposable theories, the extension into trees of first order theories, the flexible theories and the solver in T^*_{ad} are our main contributions in this thesis.

Chapter 1

Preliminaries

Contents

We present in this chapter the basic definitions of first order logic, model, theory and complete theory as well as a sufficient condition for the completeness of any first order theory.

1.1 First order language

We are given once and for all, an infinite countable set V of *variables* and the set L of *logical* symbols:

$$=, \mathit{true}, \mathit{false}, \neg, \wedge, \vee, \rightarrow, \leftrightarrow, \forall, \exists, (,).$$

We are also given once and for all, a *signature* S, i.e. a set of symbols partitioned into two subsets: the set of *function* symbols and the set of *relation* symbols. To each element s of S is linked a non-negative integer called *arity* of s. An n-ary symbol is a symbol with arity n. An 0-ary function symbol is called *constant*.

A *term* or S-*term* is word on $L \cup S \cup V$, of one of the two following forms:

$$x,\ ft_1 \dots t_n, \tag{1.1}$$

with x taken from V, f an n-ary function symbol taken from F and the t_i's shorter terms.

A *formula* or S-*formula* is word on $L \cup S \cup V$ of one of the eleven forms:

$$\begin{gathered} s = t,\ rt_1 \dots t_n,\ \mathit{true},\ \mathit{false}, \\ \neg\varphi,\ (\varphi \wedge \psi),\ (\varphi \vee \psi),\ (\varphi \rightarrow \psi),\ (\varphi \leftrightarrow \psi), \\ (\forall x\, \varphi),\ (\exists x\, \varphi), \end{gathered} \tag{1.2}$$

with s and t terms, r an n-ary relation symbol taken from S and φ and ψ shorter formulas. The set of terms and formulas forms *a first-order language with equality*.

The formulas of the first line of (1.2) are known as *atomic*, and *flat* if they are of one of the following forms:

$$true,\ false,\ x_0 = x_1, x_0 = fx_1...x_n,\ rx_1...x_n,$$

where all the x_i's are possibly non-distinct variables taken from V, f is an n-ary function symbol taken from S and r is an n-ary relation symbol taken from S. An *equation* is a formula of the form $s = t$ with s and t terms. A *relation* is a formula of the form $rt_1 \dots t_n$ with r an n-ary relation symbol taken from S and the t_i's terms.

An occurrence of a variable x in a formula is *bound* if it occurs in a sub-formula of the form $(\forall x\, \varphi)$ or $(\exists x\, \varphi)$. It is *free* in the contrary case. The *free variables of a formula* are those which have at least one free occurrence in this formula. A *proposition* or a *sentence* is a formula without free variables. If φ is a formula, then we denote by $var(\varphi)$ the set of the free variables of φ.

The syntax of the formulas being constraining, we allowed ourselves to use infix notations for the binary symbols and to add and remove brackets when there are no ambiguities.

We do not distinguish two formulas which can be made equal using the following transformations of the sub-formulas:

$$\varphi \wedge \psi \Longrightarrow \psi \wedge \varphi,\ \ (\varphi \wedge \psi) \wedge \phi \Longrightarrow \varphi \wedge (\psi \wedge \phi),$$
$$\varphi \wedge true \Longrightarrow \varphi,\ \ \varphi \vee false \Longrightarrow \varphi.$$

If I is the set $\{i_1, ..., i_n\}$, we call *conjunction* of formulas and write $\bigwedge_{i \in I} \varphi_i$, each formula of the form $\varphi_{i_1} \wedge \varphi_{i_2} \wedge ... \wedge \varphi_{i_n} \wedge true$. In particular, for $I = \emptyset$, the conjunction $\bigwedge_{i \in I} \varphi_i$ is reduced to *true*. We denote by FL the set of the conjunctions of flat formulas. We denote by AT the set of the conjunctions of atomic formulas. A set Ψ of formulas is *closed for the conjunction* if for each formula $\varphi \in \Psi$ and each formula $\phi \in \Psi$, the formula $\varphi \wedge \phi$ belongs to Ψ. All theses considerations will be useful for the algorithm of resolution given in section 4.

1.2 Model and theory

1.2.1 Model

A *model* or *S-model* is a 3-tuple $M = (\mathcal{M}, \mathcal{F}, \mathcal{R})$, where $\mathcal{M}$ is a **nonempty set** disjoint from S, its elements are called *individuals* of M ; $\mathcal{F}$ and $\mathcal{R}$ are sets of functions and relations in $\mathcal{M}$, subscripted by the elements of S. More exactly, if $\mathcal{F}$ and $\mathcal{R}$ are denoted by $(f^M)_{f \in F}$ respectively $(r^M)_{r \in R}$ then:

- $\mathcal{M}$, the *universe* or *domain* of M, is a **nonempty set** disjoint from S, its elements are called *individuals* of M;
- for every n-ary function symbol f taken from F, f^M is an n-ary operation in $\mathcal{M}$, i.e. an application from $\mathcal{M}^n$ in $\mathcal{M}$. In particular, when f is a constant, f^M belongs to $\mathcal{M}$;
- for every n-ary relation symbol r taken from R, r^M is an n-ary relation in $\mathcal{M}$, i.e. a subset of $\mathcal{M}^n$.

Let $M = (\mathcal{M}, \mathcal{F}, \mathcal{R})$ be a model. An *M-formula* φ is a formula built on the signature $S \cup \mathcal{M}$ instead of S, by considering the elements of $\mathcal{M}$ as 0-ary function symbols. If for each free variable x of φ, we replace each free occurrence of x by a same element in $\mathcal{M}$, we get an M-formula called *instantiation* or *valuation* of φ by individuals of M.

If φ is a M-formula, we say that φ *is true in* M and we write

$$M \models \varphi, \tag{1.3}$$

iff for any instantiation φ' of φ by individuals of M, the set $\mathcal{M}$ has the property expressed by $\varphi\prime$, when we interpret the function and relation symbols of $\varphi\prime$ by the corresponding functions and relations of M and when we give to the logical symbols their usual meaning.

Remark 1.2.1.1 *For every M-formula φ without free variables, one and only one of the following properties holds: $M \models \varphi$, $M \models \neg\varphi$.*

Let us finish this sub-section by a convenient notation. Let $\bar{x} = x_1...x_n$ be a word on V and let $\bar{i} = i_1...i_n$ be a word on $\mathcal{M}$ or V of the same length as $\bar{x}$. If $\varphi(\bar{x})$ and ϕ are two M-formulas, then we denote by $\varphi(\bar{i})$, respectively $\phi_{\bar{x}\leftarrow\bar{i}}$, the M-formula obtained by replacing in $\varphi(\bar{x})$, respectively in ϕ, each free occurrence of x_j by i_j

1.2.2 Theory

A *theory* is a (possibly infinite) set of propositions called *axioms*. We say that the model M is a *model of* T, iff for each element φ of T, $M \models \varphi$. If φ is a formula, we write

$$T \models \varphi,$$

iff for each model M of T, $M \models \varphi$. We say that the formulas φ and ψ are *equivalent in* T iff $T \models \varphi \leftrightarrow \psi$.

A set Ψ of formulas is called T-closed if:

- $\Psi \subseteq AT$,
- Ψ is closed for the conjunction,
- every flat formula φ is equivalent in T to a formula which belongs to Ψ and does not contain other free variables than those of φ.

The sets AT and FL are T-closed in any theory T. This notion of T-closed set is useful when we need to transform formulas of FL into formulas which belong to Ψ. The transformation of normalized formulas to working formulas defined in Section 2.3.2 illustrates this notion.

1.2.3 Complete theory

A theory T is *complete* iff for every proposition φ, one and only one of the following properties holds: $T \models \varphi$, $T \models \neg\varphi$.

Let us present now a sufficient condition for the completeness of any first-order theory. We will use the abbreviation wnfv for *"without new free variables* ". A formula φ is equivalent to a wnfv formula ψ in T means that $T \models \varphi \leftrightarrow \psi$ and ψ does not contain other free variables than those of φ.

Property 1.2.3.1 *A theory T is complete if there exists a set of formulas, called* basic formulas*, such that:*

1. *every flat formula is equivalent in T to a wnfv Boolean combination of basic formulas,*
2. *every basic formula without free variables is equivalent in T, either to true or to false,*
3. *every formula of the form*
$$\exists x\,((\textstyle\bigwedge_{i\in I}\varphi_i) \wedge (\bigwedge_{i\in I'}\neg\varphi_i)), \tag{1.4}$$
where the φ_i's are basic formulas, is equivalent in T to a wnfv Boolean combination of basic formulas.

Proof. Let Φ be the set of all the formulas which are equivalent in T to a wnfv Boolean combination of basic formulas.

Let us show first that every formula ψ belongs to Φ. Let us make a proof by induction on the syntactic structure of ψ. Without losing generalities we can restrict ourselves to the cases where ψ contains only flat formulas and the following logical symbols[1]: $\neg$, $\wedge$, $\exists$. If ψ is a flat formula, then $\psi \in \Phi$ according to the first condition of the property. If ψ is of the form $\neg\varphi_1$ or $\varphi_1 \wedge \varphi_2$, with $\varphi_1, \varphi_2 \in \Phi$, then $\psi \in \Phi$ according to the definition of Φ. If ψ is of the form $\exists x\, \varphi$, with $\varphi \in \Phi$, then according to the definition of Φ, the formula φ is equivalent to a wnfv formula φ', which is a Boolean combination of basic formulas φ_{ij}. Without losing generalities we can suppose that φ' is of the form

$$\varphi' = \bigvee_{i\in I}((\bigwedge_{j\in J} \varphi_{ij}) \wedge (\bigwedge_{j\in J'} \neg\varphi_{ij})). \tag{1.5}$$

By distributing the existential quantifier, the formula $\exists x\, \varphi'$ is equivalent in T to

$$\bigvee_{i\in I}(\exists x\, ((\bigwedge_{j\in J} \varphi_{ij}) \wedge (\bigwedge_{j\in J'} \neg\varphi_{ij}))), \tag{1.6}$$

which, according to the third condition of the property, belongs to Φ. Thus the formula $\exists x\, \varphi$, i.e. ψ, belongs to Φ.

Let now ψ be a proposition. According to what we have just shown $\psi \in \Phi$. Thus, the formula ψ is equivalent in T to a Boolean combination of basic formulas without free variables. According to the second condition of the property, one and only one of the following properties holds: $T \models \psi$, $T \models \neg\psi$. Thus T is a complete theory.□

This sufficient condition is interesting in sense that it reasons on the syntactic structure of first-order formulas and not on the semantic meaning of function and relation symbols of the theory. Informally, the basic formulas are generally formulas of the form $\exists \bar{x}\alpha$ with $\alpha \in AT$.

Corollary 1.2.3.2 *If T satisfies the three conditions of Property 1.2.3.1 then every formula is equivalent in T to a wnfv Boolean combination of basic formulas.*

This corollary is a consequence of the proof of Property 1.2.3.1 in which we have shown that if Φ is the set of all the formulas which are equivalent in T to a wnfv Boolean combination of basic formulas then every formula ψ belongs to Φ.

[1]Since each atomic formula is equivalent in the empty theory to a quantified conjunction of flat formulas and each formula is equivalent in the empty theory to a formula which contains only the logical symbols: $\exists$, $\wedge$, $\neg$.

Chapter 2

Infinite-decomposable theory

Contents

We present in this chapter a formal definition of the infinite-decomposable theories. The main idea behind this definition consists in decomposing each quantified conjunction of atomic formulas into three embedded sequences of quantifications having very particular properties, which can be expressed with the help of three special quantifiers denoted by $\exists ?$, $\exists !$, $\exists_{\infty}^{\Psi(u)}$ and called *at-most-one*, *exactly-one*, *infinite*. We show the completeness of these theories using the sufficient condition defined in Chapter 1, and give some examples of fundamental infinite-decomposable theories. We present also a decision procedure in every infnite-decomposable theory T, in the form of five rewriting rules which transform any formula φ, which can possibly contain free variables, into a wnfv conjunction ϕ of solved formulas, equivalent to φ in T and such that ϕ is, either the formula *true*, or the formula $\bigwedge_{i \in I} \neg true$, or a formula having at least one free variable and being easily transformable into a boolean combination of conjunctions of quantified atomic formulas. In particular, if φ has no free variables then ϕ is either the formula *true*, or the formula

$\neg true$. The correctness of our algorithm is another proof of the completeness of the decomposable theories. We end this chapter by an application to the theory $\mathcal{T}$ of finite or infinite trees. We show that $\mathcal{T}$ is infinite-decomposable and give two examples of solving first order propositions in $\mathcal{T}$. Note that the results presented in this chapter have been published in: [18], [19], [22].

2.1 Special quantifiers

2.1.1 Vectorial quantifiers : $\exists?$, $\exists!$

Let M be a model and let T be a theory. Let $\bar{x} = x_1 \dots x_n$ and $\bar{y} = y_1 \dots y_n$ be two words on V of the same length. Let ψ, ϕ, φ and $\varphi(\bar{x})$ be M-formulas. We write

$$\begin{array}{ll} \exists \bar{x}\, \varphi & \text{for } \exists x_1 ... \exists x_n\, \varphi, \\ \forall \bar{x}\, \varphi & \text{for } \forall x_1 ... \forall x_n\, \varphi, \\ \exists ? \bar{x}\, \varphi(\bar{x}) & \text{for } \forall \bar{x} \forall \bar{y}\, \varphi(\bar{x}) \wedge \varphi(\bar{y}) \rightarrow \bigwedge_{i \in \{1,\dots,n\}} x_i = y_i, \\ \exists ! \bar{x}\, \varphi & \text{for } (\exists \bar{x}\, \varphi) \wedge (\exists ? \bar{x}\, \varphi). \end{array}$$

The word $\bar{x}$, which can be the empty word ε, is called *vector of variables.* Note that the formulas $\exists ? \varepsilon \varphi$ and $\exists ! \varepsilon \varphi$ are respectively equivalent to *true* and to φ in any model M.

Notation 2.1.1.1 *Let Q be a vectorial quantifier taken from $\{\forall, \exists, \exists!, \exists?\}$. Let $\bar{x}$ be vector of variables taken from V. Let φ and ϕ be formulas. We write:*

$$Q\bar{x}\, \varphi \wedge \phi \;\; for \;\; Q\bar{x}\, (\varphi \wedge \phi).$$

Example 2.1.1.2 *Let $I = \{1, ..., n\}$ be a finite set. Let φ and ϕ_i with $i \in I$ be formulas. Let $\bar{x}$ and $\bar{y}_i$ with $i \in I$ be vectors of variables. We write:*

$$\begin{array}{ll} \exists \bar{x}\, \varphi \wedge \neg \phi_1 & \textit{for } \exists \bar{x}\, (\varphi \wedge \neg \phi_1), \\ \forall \bar{x}\, \varphi \wedge \phi_1 & \textit{for } \forall \bar{x}\, (\varphi \wedge \phi_1), \\ \exists ! \bar{x}\, \varphi \wedge \bigwedge_{i \in I} (\exists \bar{y}_i \phi_i) & \textit{for } \exists ! \bar{x}\, (\varphi \wedge (\exists \bar{y}_1 \phi_1) \wedge ... \wedge (\exists \bar{y}_n \phi_n) \wedge \textit{true}), \\ \exists ? \bar{x}\, \varphi \wedge \bigwedge_{i \in I} \neg (\exists \bar{y}_i \phi_i) & \textit{for } \exists ? \bar{x}\, (\varphi \wedge (\neg (\exists \bar{y}_1 \phi_1)) \wedge ... \wedge (\neg (\exists \bar{y}_n \phi_n)) \wedge \textit{true}). \end{array}$$

Property 2.1.1.3 *If $T \models \exists ? \bar{x}\, \varphi$ then*

$$T \models (\exists \bar{x}\, \varphi \wedge \neg \phi) \leftrightarrow ((\exists \bar{x} \varphi) \wedge \neg (\exists \bar{x}\, \varphi \wedge \phi)). \tag{2.1}$$

Proof. Let M be a model of T and let $\exists \bar{x}\, \varphi' \wedge \neg \phi'$ be an instantiation of $\exists \bar{x}\, \varphi \wedge \neg \phi$ by individuals of M. Let us denote by φ'_1 the M-formula $(\exists \bar{x}\, \varphi' \wedge \neg \phi')$ and by φ'_2 the M-formula $(\exists \bar{x}\, \varphi') \wedge \neg (\exists \bar{x} \varphi' \wedge \phi')$. To show the equivalence (2.1), it is enough to show that

$$M \models \varphi'_1 \leftrightarrow \varphi'_2. \tag{2.2}$$

If $M \models \neg (\exists \bar{x}\, \varphi')$ then $M \models \neg \varphi'_1$ and $M \models \neg \varphi'_2$, thus the equivalence (2.2) holds.
If $M \models \exists \bar{x}\, \varphi'$. Since $T \models \exists ? \bar{x}\, \varphi'$, there exists a unique vector $\bar{i}$ of individuals of M such that $M \models \varphi'_{\bar{x} \leftarrow \bar{i}}$. Two cases arise:

If $M \models \neg (\phi'_{\bar{x} \leftarrow \bar{i}})$, then $M \models (\varphi' \wedge \neg \phi')_{\bar{x} \leftarrow \bar{i}}$, thus $M \models \varphi'_1$. Since $\bar{i}$ is unique and since $M \models \neg (\phi'_{\bar{x} \leftarrow \bar{i}})$, there exists no vector $\bar{u}$ of individuals of M such that $M \models (\varphi' \wedge \phi')_{\bar{x} \leftarrow \bar{u}}$. Consequently, $M \models \neg (\exists \bar{x}\, \varphi' \wedge \phi')$ and thus $M \models \varphi'_2$. We have $M \models \varphi'_1$ and $M \models \varphi'_2$, thus, the equivalence (2.2) holds.

If $M \models \phi'_{\bar{x}\leftarrow\bar{i}}$, then $M \models (\varphi' \wedge \phi')_{\bar{x}\leftarrow\bar{i}}$ and thus $M \models \neg\varphi'_2$. Since $\bar{i}$ is unique and since $M \models \phi'_{\bar{x}\leftarrow\bar{i}}$, there exists no vector $\bar{u}$ of individuals of M such that $M \models (\varphi' \wedge \neg\phi')_{\bar{x}\leftarrow\bar{u}}$. Consequently, $M \models \neg(\exists\bar{x}\,\varphi' \wedge \neg\phi')$ and thus $M \models \neg\varphi'_1$. We have $M \models \neg\varphi'_1$ and $M \models \neg\varphi'_2$, thus, the equivalence (2.2) holds.□

Corollary 2.1.1.4 *If* $T \models \exists?\bar{x}\,\varphi$ *then*

$$T \models (\exists\bar{x}\,\varphi \wedge \bigwedge_{i\in I} \neg\phi_i) \leftrightarrow ((\exists\bar{x}\varphi) \wedge \bigwedge_{i\in I} \neg(\exists\bar{x}\,\varphi \wedge \phi_i)).$$

Proof. Let ψ be the formula $\neg(\bigwedge_{i\in I} \neg\phi_i)$. The formula $\exists\bar{x}\,\varphi \wedge \bigwedge_{i\in I} \neg\phi_i$, is equivalent in T to $\exists\bar{x}\,\varphi \wedge \neg\psi$. Since $T \models \exists?\bar{x}\,\varphi$, then according to Property 2.1.1.3 the preceding formula is equivalent in T to $(\exists\bar{x}\,\varphi) \wedge \neg(\exists\bar{x}\,\varphi \wedge \psi)$, which is equivalent in T to $(\exists\bar{x}\,\varphi) \wedge \neg(\exists\bar{x}\,\varphi \wedge \neg(\bigwedge_{i\in I} \neg\phi_i))$, thus to $(\exists\bar{x}\,\varphi) \wedge \neg(\exists\bar{x}\,\varphi \wedge (\bigvee_{i\in I} \phi_i))$, which is equivalent in T to $(\exists\bar{x}\,\varphi) \wedge \neg(\exists\bar{x}\,(\bigvee_{i\in I}(\varphi \wedge \phi_i)))$, thus to $(\exists\bar{x}\,\varphi) \wedge \neg(\bigvee_{i\in I}(\exists\bar{x}\,\varphi \wedge \phi_i))$, which is finally equivalent in T to

$$(\exists\bar{x}\,\varphi) \wedge \bigwedge_{i\in I} \neg(\exists\bar{x}\,\varphi \wedge \phi_i).$$

□

Property 2.1.1.5 *If* $T \models \exists?\bar{y}\phi$ *and if all the variables of* $\bar{y}$ *has no free occurrences in* φ *then*

$$T \models (\exists\bar{x}\,\varphi \wedge \neg(\exists\bar{y}\,\phi \wedge \psi)) \leftrightarrow \begin{bmatrix} (\exists\bar{x}\,\varphi \wedge \neg(\exists\bar{y}\,\phi)) \\ \vee \\ (\exists\overline{xy}\,\varphi \wedge \phi \wedge \neg\psi) \end{bmatrix}. \qquad (2.3)$$

Proof. The formula

$$\exists\bar{x}\,\varphi \wedge \neg(\exists\bar{y}\,\phi \wedge \psi),$$

is equivalent in T to

$$\exists\bar{x}\,\varphi \wedge \neg(\exists\bar{y}\,\phi \wedge \neg(\neg\psi)),$$

which according to Property 2.1.1.3 is equivalent in T to

$$\exists\bar{x}\,\varphi \wedge \neg((\exists\bar{y}\,\phi) \wedge \neg(\exists\bar{y}\phi \wedge \neg\psi)),$$

i.e. to

$$\exists\bar{x}\,\varphi \wedge ((\neg(\exists\bar{y}\,\phi)) \vee (\exists\bar{y}\phi \wedge \neg\psi)),$$

i.e. to

$$\begin{bmatrix} (\exists\bar{x}\,\varphi \wedge \neg(\exists\bar{y}\,\phi)) \\ \vee \\ (\exists\bar{x}\,\varphi \wedge (\exists\bar{y}\phi \wedge \neg\psi)) \end{bmatrix}.$$

Since all the variables of $\bar{y}$ has no free occurrences in φ, then the preceding formula is equivalent in T to

$$\begin{bmatrix} (\exists\bar{x}\,\varphi \wedge \neg(\exists\bar{y}\,\phi)) \\ \vee \\ (\exists\bar{x}\bar{y}\,\varphi \wedge \phi \wedge \neg\psi) \end{bmatrix}.$$

□

Property 2.1.1.6 *If* $T \models \exists ! \bar{x}\, \varphi$ *then*

$$T \models (\exists \bar{x}\, \varphi \wedge \neg \phi) \leftrightarrow \neg(\exists \bar{x}\, \varphi \wedge \phi).$$

Corollary 2.1.1.7 *If* $T \models \exists ! \bar{x}\, \varphi$ *then*

$$T \models (\exists \bar{x}\, \varphi \wedge \bigwedge_{i \in I} \neg \phi_i) \leftrightarrow \bigwedge_{i \in I} \neg(\exists \bar{x}\, \varphi \wedge \phi_i).$$

Corollary 2.1.1.8 *If* $T \models \psi \rightarrow (\exists ! \bar{x}\, \varphi)$ *then*

$$T \models (\psi \wedge (\exists \bar{x}\, \varphi \wedge \bigwedge_{i \in I} \neg \phi_i)) \leftrightarrow (\psi \wedge \bigwedge_{i \in I} \neg(\exists \bar{x}\, \varphi \wedge \phi_i)).$$

2.1.2 Infinite quantifier: $\exists_{\infty}^{\Psi(u)}$

Let M be a model. Let T be a theory. Let $\varphi(x)$ be a M-formula and let $\Psi(u)$ be a set of formulas having at most u as free variable.

Definition 2.1.2.1 *We write*

$$M \models \exists_{\infty}^{\Psi(u)} x\, \varphi(x), \tag{2.4}$$

iff for each instantiation $\exists x\, \varphi'(x)$ *of* $\exists x\, \varphi(x)$ *by individuals of* M *and for each finite subset* $\{\psi_1(u), .., \psi_n(u)\}$ *of elements of* $\Psi(u)$*, the set of the individuals* i *of* M *such that* $M \models \varphi'(i) \wedge \bigwedge_{j \in \{1,\dots,n\}} \neg \psi_j(i)$ *is infinite.*
We write $T \models \exists_{\infty}^{\Psi(u)} x\, \varphi(x)$*, iff for each model* M *of* T *we have (2.4).*

This infinite quantifier holds only for infinite models, i.e. models whose set of elements are infinite. Note that if $\Psi(u) = \{\mathit{false}\}$ then (2.4) simply means that M contains an infinite set of individuals i such that $\varphi(i)$. Informally, the notation (2.4) states that there exists a full elimination of quantifiers in formulas of the form $\exists x\, \varphi(x) \wedge \bigwedge_{j \in \{1,\dots,n\}} \neg \psi_j(x)$ due to an infinite set of distinct values of x in M which satisfy this formula. The intuitions behind this definition come from an aim to eliminate all the quantifiers in formulas of the form $\exists \bar{x}\, \varphi \wedge \bigwedge_{i \in I} \neg \phi_i$ where I is a finite (possibly empty) set and the ϕ_i are formulas which do not accept elimination of quantifiers. The theory of finite or infinite trees presented in Section 2.4 is a good example of theory which does not accept full elimination of quantifiers. The set $\Psi(u)$ contains in this case formulas of the form $\exists \bar{x}\, y = f(\bar{x})$ which can not be reduced anymore.

Property 2.1.2.2 *Let* J *be a finite (possibly empty) set and let* $\varphi(x)$ *and* $\varphi_j(x)$ *be* M*-formulas with* $j \in J$*. If* $T \models \exists_{\infty}^{\Psi(u)} x\, \varphi(x)$ *and if for each* $\varphi_j(x)$*, at least one of the following properties holds:*

- $T \models \exists ? x\, \varphi_j(x)$,
- *there exists* $\psi_j(u) \in \Psi(u)$ *such that* $T \models \forall x\, \varphi_j(x) \rightarrow \psi_j(x)$,

then

$$T \models \exists x\, \varphi(x) \wedge \textstyle\bigwedge_{j \in J} \neg \varphi_j(x)$$

Proof. Let M be a model of T and let $\exists x\, \varphi'(x) \wedge \bigwedge_{j\in J} \neg\varphi'_j(x)$ be an instantiation of $\exists x\, \varphi(x) \wedge \bigwedge_{j\in J} \neg\varphi_j(x)$ by individuals of M. Suppose that the conditions of Property 2.1.2.2 hold and let us show that

$$M \models \exists x\, \varphi'(x) \wedge \bigwedge_{j\in J} \neg\varphi'_j(x). \tag{2.5}$$

Let J' be the set of the $j \in J$ such that $M \models \exists ?x\, \varphi'_j(x)$ and let m be the cardinality of J'. Since for all $j \in J'$, $M \models \exists ?x\, \varphi'_j(x)$, then for each set $\mathcal{M}'$ of individuals of M such that $Cardinality(\mathcal{M}') > m$, there exists $i \in \mathcal{M}'$ such that

$$M \models \bigwedge_{j\in J'} \neg\varphi'_j(i). \tag{2.6}$$

On the other hand, since $T \models \exists_\infty^{\Psi(u)} x\, \varphi(x)$ and according to Definition 2.1.2.1 we know that for each finite subset $\{\psi_1(u), ..., \psi_n(u)\}$ of $\Psi(u)$, the set of the individuals i of M such that $M \models \varphi'(i) \wedge \bigwedge_{k=1}^{n} \neg\psi_k(i)$ is infinite. Since for all $j \in J - J'$ we have $M \models \forall x\, \varphi_j(x) \rightarrow \psi_j(x)$, thus, $M \models \forall x\, (\neg\psi_j(x)) \rightarrow (\neg\varphi_j(x))$, then there exists an infinite set ξ of individuals i of M such that $M \models \varphi'(i) \wedge \bigwedge_{j\in J-J'} \neg\varphi'_j(i)$. Since ξ is infinite then $Cardinality(\xi) > m$, and thus according to (2.6) there exists at least an individual $i \in \xi$ such that $M \models \varphi'(i) \wedge (\bigwedge_{j\in J-J'} \neg\varphi'_j(i)) \wedge (\bigwedge_{k\in J'} \neg\varphi'_k(i))$. Thus, we have

$$M \models \exists x\, \varphi'(x) \wedge \bigwedge_{j\in J} \neg\varphi'_j(x).$$

□

Property 2.1.2.3 *If $T \models \exists_\infty^{\Psi(u)} x\, \varphi(x)$ then $T \models \exists_\infty^{\Psi(u)} x\, true$.*

Proof. Let M be a model of T. If $T \models \exists_\infty^{\Psi(u)} x\, \varphi(x)$ then $M \models \exists_\infty^{\Psi(u)} x\, \varphi(x)$. According to Definition 2.1.2.1 there exists an infinite set of individuals i such that $M \models \varphi(i) \wedge \bigwedge_{j\in J} \neg\varphi_j(i)$ with $\varphi_j(u) \in \Psi(u)$ for all $j \in J$. Thus there exists an infinite set of individuals i such that $M \models true \wedge \bigwedge_{j\in J} \neg\varphi_j(i)$, i.e. $M \models \exists_\infty^{\Psi(u)} x\, true$ and thus $T \models \exists_\infty^{\Psi(u)} x\, true$. □

2.2 Infinite-decomposable theory

2.2.1 Definition

Definition 2.2.1.1 *A theory T having at least one model is called* infnite-decomposable *or quit simply* decomposable, *if there exists a set $\Psi(u)$ of formulas having at most u as free variable, a T-closed set A and three sets A', A'' and A''' of formulas of the form $\exists \bar{x}\alpha$ with $\alpha \in A$ such that:*

1. *Every formula of the form $\exists \bar{x}\, \alpha \wedge \psi$, with $\alpha \in A$ and ψ any formula, is equivalent in T to a wnfv decomposed formula of the form*

$$\exists \bar{x}'\, \alpha' \wedge (\exists \bar{x}''\, \alpha'' \wedge (\exists \bar{x}'''\, \alpha''' \wedge \psi)),$$

with $\exists \bar{x}'\, \alpha' \in A'$, $\exists \bar{x}''\, \alpha'' \in A''$ and $\exists \bar{x}'''\, \alpha''' \in A'''$.

2. *If $\exists \bar{x}'\alpha' \in A'$ then $T \models \exists ?\bar{x}'\, \alpha'$ and for each free variable y in $\exists \bar{x}'\alpha'$, at least one of the following properties holds:*

- *$T \models \exists ? y \bar{x}' \, \alpha'$,*
- *there exists $\psi(u) \in \Psi(u)$ such that $T \models \forall y \, (\exists \bar{x}' \, \alpha') \rightarrow \psi(y)$.*

3. *If $\exists \bar{x}'' \alpha'' \in A''$ then for each x''_i of $\bar{x}''$ we have $T \models \exists^{\Psi(u)}_{\infty} x''_i \, \alpha''$.*
4. *If $\exists \bar{x}''' \alpha''' \in A'''$ then $T \models \exists ! \bar{x}''' \, \alpha'''$.*
5. *If the formula $\exists \bar{x}' \alpha'$ belongs to A' and has no free variables then this formula is either the formula $\exists \varepsilon true$ or $\exists \varepsilon false$.*

Since A is T-closed, then A is a sub-set of AT. While the formulas of A'' and A''' accept full elimination of quantifiers according to the properties of the quantifiers $\exists !$ and $\exists^{\Psi(u)}_{\infty}$, the formulas of A' can possibly do not accept elimination of quantifiers. This is due to the second point of Definition 2.2.1.1 which states that $T \models \exists ? \bar{x}' \alpha'$. The computation of the sets A, A', A'', A''' and $\Psi(u)$ for a theory T depends on the axiomatization of T. Generally, it is enough to know solving a formula of the form $\exists \bar{x} \alpha$ with $\alpha \in FL$ to get a first intuition on the sets A', A'', A''' and $\Psi(u)$. Informally, the sets A', A'' and A''' can be called according to their linked vectorial quantifier, i.e. A' is the *at most one solution set* and contains formulas which accept at most one solution in T and possibly do not accept full elimination of quantifiers, the set A'' is the *infinite instantiation set* and contains formulas that accept an infinite set of solutions in T. The set A''' is the *unique solution set* and contains formulas which have one and only solution in T. The set $\Psi(u)$ contains generally simple formulas of the form $\exists \bar{x} \alpha$ with at most one free variable and $\alpha \in A$. It can also be reduced for example to the set $\{faux\}$. Note that the sets A' and A''' are generally not empty since for each model M of any theory T we have $M \models \exists ? \varepsilon \, x = y$ and $M \models \exists ! x \, x = y$.

Property 2.2.1.2 *Let T be a decomposable theory. Every formula of the form $\exists \bar{x} \, \alpha$, with $\alpha \in A$, is equivalent in T to a wnfv formula of the form $\exists \bar{x}' \, \alpha'$ with $\exists \bar{x}' \alpha' \in A'$.*

Proof. Let $\exists \bar{x} \, \alpha$ be a formula with $\alpha \in A$. According to Definition 2.2.1.1 this formula is equivalent in T to a wnfv formula of the form

$$\exists \bar{x}' \, \alpha' \wedge (\exists \bar{x}'' \, \alpha'' \wedge (\exists \bar{x}''' \, \alpha''')),$$

with $\exists \bar{x}' \, \alpha' \in A'$, $\exists \bar{x}'' \, \alpha'' \in A''$ and $\exists \bar{x}''' \, \alpha''' \in A'''$. Since $\exists \bar{x}''' \, \alpha''' \in A'''$ then according to Definition 2.2.1.1 $T \models \exists ! \bar{x}''' \alpha'''$ and thus using Corollary 2.1.1.7 (with $\phi = \mathit{false}$) the preceding formula is equivalent in T to

$$\exists \bar{x}' \, \alpha' \wedge (\exists \bar{x}'' \, \alpha''),$$

which is equivalent in T to

$$\exists \bar{x}' \, \alpha' \wedge (\exists x''_1 ... x''_{n-1} \, (\exists x''_n \, \alpha'')).$$

Since $\exists \bar{x}'' \, \alpha'' \in A''$ then according to Definition 2.2.1.1 we have $T \models \exists^{\Psi(u)}_{\infty} x''_n \, \alpha''$ and thus $T \models \exists \, x''_n \, \alpha''$. The preceding formula is equivalent in T to

$$\exists \bar{x}' \, \alpha' \wedge (\exists x''_1 ... x''_{n-1} \, true),$$

which is finally equivalent in T to

$$\exists \bar{x}' \, \alpha'.$$

□

Using Property 2.2.1.2 and the fifth point of Definition 2.2.1.1 we get

Corollary 2.2.1.3 *Let T be a decomposable theory. Every formula, without free variables, of the form $\exists \bar{x}\, \alpha$, with $\alpha \in A$, is equivalent in T either to true or to false.*

Proof. Let $\exists \bar{x}\, \alpha$ be a proposition with $\alpha \in A$. According to Property 2.2.1.2, this proposition is equivalent in T to a proposition of the form $\exists \bar{x}'\, \alpha'$ which belongs to A'. According to the last point of Definition 2.2.1.1, this proposition is of the form $\exists \varepsilon\, true$ or $\exists \varepsilon\, false$. Since T has at least one model, then either $T \models \exists \bar{x}\, \alpha$, or $T \models \neg(\exists \bar{x}\, \alpha)$. The condition that T has at least a mode is vital ! In fact, if T has no models then we can have $T \models true \leftrightarrow false$ and thus we have both $T \models \exists \bar{x}\, \alpha$ and $T \models \neg(\exists \bar{x}\, \alpha)$. □

2.2.2 Completeness

Theorem 2.2.2.1 *If T is decomposable then T is complete.*

Proof. Let T be a decomposable theory which satisfies the five conditions of Definition 2.2.1.1. Let us show that T is complete using Property 1.2.3.1 and by taking formulas of the form $\exists \bar{x}\, \alpha$, with $\alpha \in A$, as basic formulas. Note that according to Definition 2.2.1.1, the sets A', A'' and A''' contain formulas of the form $\exists \bar{x} \alpha$ with $\alpha \in A$.

Let us show that the first condition of Property 1.2.3.1 holds, i.e. every flat formula is equivalent to a Boolean combination of basic formulas. According to Definition 2.2.1.1 the set A is T-closed, i.e. (i) A is closed for the conjunction, (ii) every flat formula is equivalent in T to a formula which belongs to A. Let α be a flat formula. According to (ii) α is equivalent in T to a formula β which belongs to A. Since β is equivalent in T to $\exists \varepsilon\, \beta$ and $\beta \in A$ then α is equivalent to a basic formula[2]. Thus, the first condition of Property 1.2.3.1 holds.

Let us show that the second condition of Property 1.2.3.1 holds, i.e. every basic formula without free variables is either equivalent to *true* or to *false* in T. Let $\exists \bar{x}\, \alpha$ with $\alpha \in A$ be a basic formula without free variables. According to Corollary 2.2.1.3 either $T \models \exists \bar{x} \alpha$ or $T \models \neg(\exists \bar{x}\, \alpha)$. Thus, the second condition of Property 1.2.3.1 holds.

Let us show now that the third condition of Property 1.2.3.1 holds, i.e. every formula of the form

$$\exists x\, (\bigwedge_{i \in I} (\exists \bar{x}_i\, \alpha_i)) \wedge (\bigwedge_{j \in J} \neg(\exists \bar{y}_j\, \beta_j)), \tag{2.7}$$

with $\alpha_i \in A$ for all $i \in I$ and $\beta_j \in A$ for all $j \in J$, is equivalent in T to a wnfv Boolean combination of basic formulas, i.e. to a wnfv Boolean combination of formulas of the form $\exists \bar{x} \alpha$ with $\alpha \in A$. By lifting all the quantifications $\exists \bar{x}_i$ after having possibly renamed the variables which appear in each $\bar{x}_i$, the formula (2.7) is equivalent in T to a wnfv formula of the form

$$\exists \bar{x}\, (\bigwedge_{i \in I} \alpha_i) \wedge \bigwedge_{j \in J} \neg(\exists \bar{y}_j\, \beta_j),$$

with $\alpha_i \in A$ for all $i \in I$ and $\beta_j \in A$ for all $j \in J$. According to Definition 2.2.1.1 the set A is T-closed and thus closed under conjunction. The preceding formula is equivalent in T to a wnfv formula of the form

$$\exists \bar{x}\, \alpha \wedge \bigwedge_{j \in J} \neg(\exists \bar{y}_j\, \beta_j),$$

with $\alpha \in A$ and $\beta_j \in A$ for all $j \in J$. According to the first point of Definition 2.2.1.1 the preceding formula is equivalent in T to a wnfv formula of the form

$$\exists \bar{x}'\, \alpha' \wedge (\exists \bar{x}''\, \alpha'' \wedge (\exists \bar{x}'''\, \alpha''' \wedge \bigwedge_{j \in J} \neg(\exists \bar{y}_j\, \beta_j))),$$

[2] Of course a basic formula is a particular case of a Boolean combination of basic formulas.

with $\exists \bar{x}' \, \alpha' \in A'$, $\exists \bar{x}'' \, \alpha'' \in A''$, $\exists \bar{x}''' \, \alpha''' \in A'''$ and $\beta_j \in A$ for all $j \in J$. Since $\exists \bar{x}''' \, \alpha''' \in A'''$ then according to the fourth point of Definition 2.2.1.1 $T \models \exists ! \bar{x}''' \, \alpha''$. Thus, using Corollary 2.1.1.7 the preceding formula is equivalent in T to

$$\exists \bar{x}' \, \alpha' \wedge (\exists \bar{x}'' \, \alpha'' \wedge \bigwedge_{j \in J} \neg(\exists \bar{x}''' \, \alpha''' \wedge (\exists \bar{y}_j \, \beta_j))).$$

By lifting all the quantifies $\exists \bar{y}_j$ after having possibly renamed the variables which appear in each $\bar{y}_j$, the preceding formula is equivalent in T to

$$\exists \bar{x}' \, \alpha' \wedge (\exists \bar{x}'' \, \alpha'' \wedge \bigwedge_{j \in J} \neg(\exists \bar{x}''' \exists \bar{y}_j \, \alpha''' \wedge \beta_j)).$$

According to Definition 2.2.1.1 the sets A', A'' and A''' contain formulas of the form $\exists \bar{x} \alpha$ with $\alpha \in A$, thus $\alpha''' \in A$. Since $\beta_j \in A$ for all $j \in J$ and since A is T-closed (i.e. closed for the conjunction...) then for all $j \in J$ the formula $\alpha''' \wedge \beta_j$ belongs to A. Thus, the preceding formula is equivalent in T to a wnfv formula of the form

$$\exists \bar{x}' \, \alpha' \wedge (\exists \bar{x}'' \, \alpha'' \wedge \bigwedge_{j \in J} \neg(\exists \bar{y}_j \, \beta_j)),$$

with $\exists \bar{x}' \, \alpha' \in A'$, $\exists \bar{x}'' \, \alpha'' \in A''$, and $\beta_j \in A$ for all $j \in J$. According to Corollary 2.2.1.2 the preceding formula is equivalent in T to a wnfv formula of the form

$$\exists \bar{x}' \, \alpha' \wedge (\exists \bar{x}'' \, \alpha'' \wedge \bigwedge_{j \in J} \neg(\exists \bar{y}'_j \, \beta'_j)),$$

with $\exists \bar{x}' \, \alpha' \in A'$, $\exists \bar{x}'' \, \alpha'' \in A''$, and $\exists \bar{y}'_j \, \beta'_j \in A'$ for all $j \in J$. Let us denote by J_1, the set of the $j \in J$ such that x''_n does not have free occurrences in the formula $\exists \bar{y}'_j \beta'_j$. Thus, the preceding formula is equivalent in T to

$$\exists \bar{x}' \, \alpha' \wedge (\exists x''_1 ... \exists x''_{n-1} \begin{bmatrix} (\bigwedge_{j \in J_1} \neg(\exists \bar{y}'_j \, \beta'_j)) \wedge \\ (\exists x''_n \, \alpha'' \wedge \bigwedge_{j \in J - J_1} \neg(\exists \bar{y}'_j \, \beta'_j)) \end{bmatrix}). \tag{2.8}$$

Since $\exists \bar{x}'' \, \alpha'' \in A''$ and $\exists \bar{y}'_j \, \beta'_j \in A'$, then according to Property 2.1.2.2 and the points 2 and 3 of Definition 2.2.1.1, the formula (2.8) is equivalent in T to

$$\exists \bar{x}' \, \alpha' \wedge (\exists x''_1 ... \exists x''_{n-1} \, (true \wedge \bigwedge_{j \in J_1} \neg(\exists \bar{y}'_j \, \beta'_j))).$$

By repeating the three preceding steps $(n-1)$ times, by denoting by J_k the set of the $j \in J_{k-1}$ such that $x''_{(n-k+1)}$ does not have free occurrences in $\exists \bar{y}'_j \, \beta'_j$, and by using $(n-1)$ times Property 2.1.2.3, the preceding formula is equivalent in T to

$$\exists \bar{x}' \, \alpha' \wedge \bigwedge_{j \in J_n} \neg(\exists \bar{y}'_j \, \beta'_j).$$

Since $\exists \bar{x}' \, \alpha' \in A'$ then according to the second point of Definition 2.2.1.1 $T \models \exists ? \bar{x}' \, \alpha'$. Thus, using Corollary 2.1.1.4 the preceding formula is equivalent in T to

$$(\exists \bar{x}' \, \alpha') \wedge \bigwedge_{j \in J_n} \neg(\exists \bar{x}' \, \alpha' \wedge (\exists \bar{y}'_j \, \beta'_j)).$$

By lifting all the quantifies $\exists \bar{y}_j$ after having possibly renamed the variables which appear in each $\bar{y}_j$, the preceding formula is equivalent in T to

$$(\exists \bar{x}' \, \alpha') \wedge \bigwedge_{j \in J_n} \neg(\exists \bar{x}' \exists \bar{y}'_j \, \alpha' \wedge \beta'_j).$$

According to Definition 2.2.1.1 the sets A', A'' and A''' contain formulas of the form $\exists\bar{x}\alpha$ with $\alpha \in A$. Thus, since $\exists\bar{x}'\,\alpha' \in A'$ and $\exists\bar{y}'_j\,\beta'_j \in A'$ for all $j \in J_n$, then $\alpha' \in A$ and $\beta_j \in A$ for all $j \in J_n$. Since the set A is T-closed, it is closed for the conjunction, then for all $j \in J_n$ the formula $\alpha' \wedge \beta'_j$ belongs to A and thus, the preceding formula is equivalent in T a wnfv formula of the form

$$(\exists\bar{x}\,\alpha) \wedge \textstyle\bigwedge_{j\in J_n} \neg(\exists\bar{y}_j\beta_j),$$

with $\alpha \in A$ and $\beta_j \in A$ for all $j \in J_n$. This formula is a Boolean combination of formulas of the form $\exists\bar{x}\alpha$ with $\alpha \in A$, i.e. a Boolean combination of basic formulas. Note that we have never added new free variables and we have renamed only the quantified variables. Thus, the third condition of Property 1.2.3.1 holds.

Since T satisfies the three conditions of Property 1.2.3.1, then T is a complete theory. □

According to Theorem 2.2.2.1 and Corollary 1.2.3.2, we have the following corollary:

Corollary 2.2.2.2 *If T is decomposable and if for all $\exists\bar{x}'\alpha' \in A'$ we have $\bar{x}' = \varepsilon$, then T accepts full elimination of quantifiers.*

Proof. Let T a decomposable theory such that for all $\exists\bar{x}'\alpha' \in A'$ we have $\bar{x}' = \varepsilon$. Let φ be a formula which can possibly contain free variables. In the proof of Theorem 2.2.2.1 we have shown that T satisfies the three conditions of Property 1.2.3.1 using formulas of the forms $\exists\bar{x}\alpha$ with $\alpha \in A$ as basic formulas. Thus, according to Corollary 1.2.3.2, the formula φ is equivalent in T to a wnfv Boolean combination of basic formulas, i.e. Boolean combination of formulas of the form $\exists\bar{x}\alpha$ with $\alpha \in A$. According to Property 2.2.1.2 each of these basic formulas is equivalent in T to a wnfv formula of the form $\exists\bar{x}'\alpha'$ which belongs to A'. Since for all $\exists\bar{x}'\alpha' \in A'$ we have $\bar{x}' = \varepsilon$ and since $\alpha' \in A$ (according to the structure of the set A' defined in Definition 2.2.1.1) then the formula φ is equivalent in T to a boolean combination of elements of A. Since T is decomposable then A is a T-closed set and thus $A \subseteq AT$. Then, the formula φ is equivalent in T to a wnfv boolean combination ϕ of conjunctions of atomic formulas. According to the syntax of the atomic formulas defined in Section 2, it is clear that ϕ does not contain quantifiers. □

This corollary makes the connection between the set A' and the notion of full elimination of quantifiers. In fact, if T is decomposable and does not accept full elimination of quantifiers then it is enough to add axioms to T which enable to eliminate all the quantifications of the formulas of A' to get a full elimination quantifiers theory. The sets A'' and A''' are not concerned by this notion since in any decomposable theory T the formulas of A'' and A''' accept full elimination of quantifiers due to their associated vectorial quantifiers: $\exists!$ and $\exists^{\Psi(u)}_{\infty}$. On the other hand, if T is a decomposable theory which satisfies Corollary 2.2.2.2 then we can interest ourself to get the smallest sub-set T^* of axioms of T, such that T^* still accepts full elimination of quantifiers. For that is is enough to remove axiom by axiom from T and check each time if the theory still satisfies Corollary 2.2.2.2. This corollary shows also the fact that a decomposable theory T does not means that T admits full elimination of quantifiers. In fact, the theories of infinite trees, finite trees and finite or infinite trees as defined by M. Maher [33] do not accept full elimination of quantifier but are decomposable and thus complete [19].

2.2.3 Fundamental examples

We present in this sub-section two examples of simples decomposable theories. The first one is a simple axiomatization of an infinite set of distinct individuals with an empty set of function and relation symbols. This theory denoted by Eq can be seen as a small extension of the Clark equational theory CET [3], even if according to our syntax the equality symbol is considered as a primitive logical symbol together with its usual properties (commutativity, transitivity ...). The second theory is the theory of additive rational or real numbers with addition and subtraction. The goal of these examples is to show the decomposability of simple theories whose properties are well known and do not need addition of proofs. An other example of a non-simple decomposable theory (finite or infinite trees) is given in Section 2.4 with a detailed study of the properties of this theory.

Let us assume for all this sub-section that the variables of V are ordered by a strict linear dense order relation without endpoints denoted by $\succ$.

Equality theory

Let Eq be a theory together with an empty set of function and relation symbols and whose axioms is the infinite set of propositions of the following form

$$(1_n)\ \ \forall x_1 ... \forall x_n \exists y\, \neg(x_1 = y) \wedge ... \wedge \neg(x_n = y), \tag{2.9}$$

where all the variables $x_1...x_n$ are distinct and $(n \neq 0)$. The form (2.9) is called *diagram of axiom* and for each value of n there exists *an axiom* of Eq. For example the following property is true in Eq:

$$Eq \models \exists x\, \neg(x = y) \wedge \neg(x = z).$$

The theory Eq has as model an infinite set of distinct individuals.

Note that since Eq has an empty set of function and relation symbols, then $AT = FL$ and thus all the equations of Eq are flat equations. Let x and y be two distinct variables. We call *leader* of the equation $x = y$ the variable x. A conjunction α of flat formulas is called $(\succ)$-*solved* in Eq if: (1) *false* is not a sub-formula of α, (2) all the equations of α are of the form $x = y$ with[3] $x \succ y$, (3) each equation of α has a distinct leader which does not occur in the other equations of α.

Property 2.2.3.1 *Every conjunction of flat formulas is equivalent in Eq either to false or to a* $(\succ)$-*solved conjunction of equations.*

Let x, y and z be variables such that $x \succ y \succ z$. The conjunction $x = x \wedge y = z$ is not $(\succ)$-solved because in the equation $x = x$ we have $x \not\succ x$. By the same way, the conjunction $x = y \wedge y = z$ is not $(\succ)$-solved because y is leader in $y = z$ and occurs also in $x = y$. The conjunctions *true* and $x = z \wedge y = z$ are $(\succ)$-solved. The computation of a possibly $(\succ)$-solved conjunction of equations from a conjunction of flat formulas in Eq is evident[4] and proceeds using the usual properties of the equality (commutativity, substitution, transitivity...) and by replacing each formula of the form $x = x$ and $\alpha \wedge$ *false* by *true* respectively by *false*.

Property 2.2.3.2 *Let* α *be a* $(\succ)$-*solved conjunction of equations and* $\bar{x}$ *the vector of the leaders of the equations of* α. *We have:*

1. $Eq \models \exists! \bar{x}\, \alpha$.

[3] Recall that $\succ$ is a strict linear dense order relation and thus $x \not\succ x$. In other terms $x = x$ is not $(\succ)$-solved.

[4]

(1) $y = x \Longrightarrow x = y$. (2) $x = y \wedge x = z \Longrightarrow x = y \wedge z = y$. (3) $x = y \wedge z = x \Longrightarrow x = y \wedge z = y$.
(4) $false \wedge \alpha \Longrightarrow false$. (5) $x = x \Longrightarrow true$.

The rules (1), (2) and (3) are applied only if $x \succ y$.

2. *$Eq \models \exists_{\infty}^{\{faux\}} x\, true$.*

3. *For all $x \in var(\alpha)$ we have $Eq \models \exists ? x\, \alpha$.*

The first point holds because all the leaders of the equations of α are distinct and have one and only occurrence in α. Thus, for each instantiation of the right hand sides of each equation, there exists one and only one value for the left hand sides and thus for the leaders. The second point is a consequence of the diagram of axiom (2.9) which states that for every finite set of distinct variables $x_1 ... x_n$ there exists a variable y which is different from all the x_i. Thus, in each model of Eq there exists an infinite set of individuals. Thus according to Definition 2.1.2.1 we have $Eq \models \exists_{\infty}^{\{faux\}} x\, true$. The third point holds since in a $(\succ)$-solved conjunction of equations we have no formulas of the form $x = x$ (because $x \not\succ x$). Thus, using the properties of the equality for each model of Eq and for each instantiation of the variables of $var(\alpha) - \{x\}$ either there exists a unique solution of x or there exists a contradiction in the instantiations and thus there exists no values for x.

Property 2.2.3.3 *The theory Eq is decomposable.*

Proof. We show that Eq satisfies the conditions of Definition 2.2.1.1. The sets A, A', A'', A''' and $\Psi(u)$ are chosen as follows:

- A is the set FL.
- A' is the set of formulas of the form $\exists \varepsilon\, \alpha'$ where α' is either a $(\succ)$-solved conjunction of equations or the formula *false*.
- A'' is the set of formulas of the form $\exists \bar{x}''\, true$.
- A''' is the set of formulas of the form $\exists \bar{x}''' \alpha'''$ with α''' a $(\succ)$-solved conjunction of equations and $\bar{x}'''$ the vector of the leaders of the equations of α'''.
- $\Psi(u) = \{false\}$.

It is obvious that FL is Eq-closed and A', A'' and A''' contain formulas of the form $\exists \bar{x}\, \alpha$ with $\alpha \in FL$.

Let us show that Eq satisfies the first condition of Definition 2.2.1.1. Let $\alpha \in FL$ and ψ a formula. Let $\bar{x}$ be a vector of variables. Let us choose an order $\succ$ such that the variables of $\bar{x}$ are greater than the free variables of $\exists \bar{x}\, \alpha$. According to Property 2.2.3.1 two cases arise:

Either α is equivalent to *false* in Eq and thus the formula $\exists \bar{x} \alpha \wedge \psi$ is equivalent in Eq to a decomposed formula of the form

$$\exists \varepsilon\, false \wedge (\exists \varepsilon\, true \wedge (\exists \varepsilon\, true \wedge \psi)).$$

Or, α is equivalent in Eq to a $(\succ)$-solved conjunction β of flat formulas. Let X_l be the set of the variables of $\bar{x}$ which are leader in the equations of β. Let X_n be the set of the variables of $\bar{x}$ which are not leader in the equations of β. The formula $\exists \bar{x} \alpha \wedge \psi$ is equivalent in Eq to a decomposed formula of the form

$$\exists \bar{x}'\, \alpha' \wedge (\exists \bar{x}''\, \alpha'' \wedge (\exists \bar{x}'''\, \alpha''' \wedge \psi)), \tag{2.10}$$

with $\bar{x}' = \varepsilon$. The formula α' contains the conjunction of the equations of β whose leaders do not belong to X_l, i.e. whose leaders are free in $\exists \bar{x} \beta$. The vector $\bar{x}''$ contains the variables of X_n. The formula α'' is the formula *true*. The vector $\bar{x}'''$ contains the variables of X_l. The formula α''' is the conjunction of the equations of β whose leaders belong to X_l. According to our construction it is clear that $\exists \bar{x}'\alpha' \in A'$, $\exists \bar{x}''\alpha'' \in A''$ and $\exists \bar{x}'''\alpha \in A'''$. Let us show that (2.10) and $\exists \bar{x}\alpha \wedge \psi$ are equivalent in Eq. Let X, X', X'' and X''' be the sets of the variables of the vectors[5] $\bar{x}$, $\bar{x}'$, $\bar{x}''$ and $\bar{x}'''$. If α is equivalent to *false* in Eq then the equivalence of the decomposition is evident. Else β is a $(\succ)$-solved conjunction of equations and thus according to our construction we have: $X = X' \cup X'' \cup X'''$, $X' \cap X'' = \emptyset$, $X' \cap X''' = \emptyset$, $X'' \cap X''' = \emptyset$, $X' = \emptyset$, for all $x''_i \in X''$ we have $x''_i \notin var(\alpha')$ and for all $x'''_i \in X'''$ we have $x'''_i \notin var(\alpha' \wedge \alpha'')$. This is due to the definition of the $(\succ)$-solved conjunction of flat formulas and the order $\succ$ which has been chosen such that the quantified variables of $\exists \bar{x}\, \alpha$ are greater than the free variables of $\exists \bar{x}\, \alpha$. On the other hand, each equation in β occurs in $\alpha' \wedge \alpha'' \wedge \alpha'''$ and each equation in $\alpha' \wedge \alpha'' \wedge \alpha'''$ occurs in β and thus $Eq \models \beta \leftrightarrow (\alpha' \wedge \alpha'' \wedge \alpha''')$. We have shown that the vectorial quantifications are coherent and the equivalence $Eq \models \beta \leftrightarrow \alpha' \wedge \alpha'' \wedge \alpha'''$ holds. According to Property 2.2.3.1 we have $Eq \models \alpha \leftrightarrow \beta$ and thus, the decomposition keeps the equivalence in Eq. Let us decompose for example

$$\exists xyz\, v = w \wedge z = z \wedge z = x \wedge v = y.$$

Let us choose the order $\succ$ such that $x \succ y \succ z \succ v \succ w$. Let us $(\succ)$-solve the conjunction $v = w \wedge z = z \wedge z = x \wedge v = y$. Thus the preceding formula is equivalent in Eq to

$$\exists xyz\, v = w \wedge x = z \wedge y = w.$$

Finally this formula is equivalent to the following decomposed formula

$$\exists \varepsilon\, v = w \wedge (\exists z\, true \wedge (\exists xy\, x = z \wedge y = w)).$$

The theory Eq satisfies the second condition of Definition 2.2.1.1 according to the third point of Property 2.2.3.2 and using the fact that $\bar{x}' = \varepsilon$. The theory Eq satisfies the third condition of Definition 2.2.1.1 according to the second point of Property 2.2.3.2. The theory Eq satisfies the fourth condition of Definition 2.2.1.1 according to the first point of Property 2.2.3.2. The theory Eq satisfies the last condition of Definition 2.2.1.1 because A' is of the form $\exists \varepsilon\, \alpha'$ where α' is either the formula *false* or a $(\succ)$-solved conjunction of equations. Thus, if $\exists \varepsilon\, \alpha'$ has no free variables, then either $\alpha' = true$ or $\alpha' = false$. □

Note that Eq accepts full elimination of quantifiers. In fact Corollary 2.2.2.2 illustrates this result since for all $\exists \bar{x}'\alpha' \in A'$ we have $\bar{x}' = \varepsilon$.

Additive rational or real numbers theory

Let $F = \{+, -, 0, 1\}$ a set of function symbols of respective arities $2, 1, 0, 0$. Let $R = \emptyset$ an empty set of relation symbols. Let Ra be the theory of additive rational or real numbers together with addition and subtraction. Let a be a positive integer and $t_1, ..., t_n$ terms.

Notation 2.2.3.4 *We denote by:*

- *Z the set of the integers.*
- *$t_1 + t_2$, the term $+t_1t_2$.*
- *$t_1 + t_2 + t_3$, the term $+t_1(+t_2t_3)$.*

[5]Of course if $\bar{x} = \varepsilon$ then $X = \emptyset$

- $-a.t_1$, *the term* $\underbrace{(-t_1)+\cdots+(-t_1)}_{a}$.
- $0.t_1$, *the term 0.*
- $a.t_1$, *the term* $\underbrace{t_1+\cdots+t_1}_{a}$,
- $\sum_{i=1}^{n} t_i$, *the term* $\overline{t_1+t_2+...+t_n}+0$, *where* $\overline{t_1+t_2+...+t_n}$ *is the term* $t_1+t_2+...+t_n$ *in which we have removed all the* t_i*'s which are equal to* 0. *For* $n=0$ *the term* $\sum_{i=1}^{n} t_i$ *is reduced to the term* 0.

The axiomatization of Ra is the set of propositions of one of the 8 following forms:

1	$\forall x \forall y\, x+y=y+x$,
2	$\forall x \forall y \forall z\, x+(y+z)=(x+y)+z$,
3	$\forall x\, x+0=x$,
4	$\forall x\, x+(-x)=0$,
5_n	$\forall x\, n.x=0 \rightarrow x=0$,
6_n	$\forall x\, \exists! y\, n.y=x$,
7	$\forall x \forall y \forall z\, (x=y) \leftrightarrow (x+z=y+z)$,
8	$\neg(0=1)$.

with n an non-null integer. This theory has two usual models: rational numbers Q with addition and subtraction in Q and real numbers R with addition and subtraction in R.

We call *block* every conjunction α of formulas of the form: *true*, *false*, $\sum_{i=1}^{n} a_i.x_i = a_0.1$ with $x_1,...,x_n$ distinct variables and $a_i \in Z$ for all $i \in \{0,1,...,n\}$. We call leader of an equation of the form $\sum_{i=1}^{n} a_i.x_i = a_0.1$ the greatest variables x_k ($k \in \{1,...,n\}$) according to the order $\succ$ such that $a_k \neq 0$. A block α is called ($\succ$)-*solved* in Ra if (1) each equation of α has a distinct leader which does not occur in the other equations of α and (2) α does not contain sub-formulas of the form $0 = a_0.1$ or *false* with $a_0 \in Z$. According to the axiomatization of Ra we show easily that:

Property 2.2.3.5 *For all* $k \in \{1,...,n\}$ *we have:*

$$Ra \models \sum_{i=1}^{n} a_i.x_i = a_0.1 \leftrightarrow a_k.x_k = \sum_{i=1, i\neq k}^{n} (-a_i).x_i + a_0.1$$

Property 2.2.3.6 *Every block is equivalent in Ra either to false or to a* ($\succ$)*-solved block.*

Let x, y and z be variables such that $x \succ y \succ z$. The block $2.x+y=(-1).1 \wedge 2.z+y=3.1$ is not ($\succ$)-solved because y is leader in the second equation and occurs also in the first one. By the same way, the block $x+y=3.1 \wedge 0=0.1$ is not ($\succ$)-solved because $0=0.1$ occurs in it. The blocks *true* and $x+2.z=2.1 \wedge 3.y+2.z=3.1$ are ($\succ$)-solved. The computation of a possibly ($\succ$)-solved block is evident[6] and proceeds using Property 2.2.3.5 and a usual technique of substitution and simplification by replacing each equation of the form $0=a_0.1$ by *false* if $a_0 \neq 0$ and by *true* otherwise and every formula of the form *false* $\wedge\, \alpha$ by *false*.

[6]
(1) $0=0.1 \Longrightarrow$ *true*. (2) $0=a_0.1 \Longrightarrow$ *false*. (3) *false* $\wedge\, \alpha \Longrightarrow$ *false*.

$$(4)\ \begin{bmatrix} \sum_{i=1}^{n} a_i.x_i = a_0.1 \wedge \\ \sum_{i=1}^{h} b_i.x_i = b_0.1 \end{bmatrix} \Longrightarrow \begin{bmatrix} \sum_{i=1}^{n} a_i.x_i = a_0.1 \wedge \\ \sum_{i=1}^{h} (b_k a_i - a_k b_i).x_i = (b_k a_0 - a_k b_0).1 \end{bmatrix}.$$

In the rule (2) $a_0 \neq 0$. In the rule (4) x_k is the leader of the block $\sum_{i=1}^{n} a_i.x_i = a_0.1$ and $b_k \neq 0$.

Property 2.2.3.7 *Let α be a $(\succ)$-solved block and $\bar{x}$ be the vector of the leaders of the equations of α. We have:*

1. $Ra \models \exists! \bar{x}\, \alpha$.
2. $Ra \models \exists_{\infty}^{\{faux\}} x\, true$.
3. *For all $x \in var(\alpha)$ we have $Ra \models \exists ? x\, \alpha$.*

The first point holds because all the leaders are distinct and do not occur in the other equations. Thus, if we transform each equation of the form $\sum_{i=1}^{n} a_i.x_i = a_0.1$ using Property 2.2.3.5 into a formula of the form $a_k.x_k = \sum_{i=1}^{n}(-a_i).x_i + a_0.1$ with x_k the leader of this equation, then we get a conjunction of equations whose left hand sides are distinct and do not occur in the right hand sides. Thus, for each instantiation of the right hand sides of these equations there exists one and only value for the left hand sides and thus for the leaders according to Axiom 6 of Ra. The second point holds because according to the axiom 8 we have $Ra \models \neg(0 = 1)$ thus using the axiom 7 we have $Ra \models \neg(0 + 1 = 1 + 1)$. Then using the axiom 3 we get $Ra \models \neg(1 = 1 + 1)$. Thus using the transitivity of the equality we have $Ra \models \neg(0 = 1 + 1)$. If we repeat the preceding steps n times we get n different individuals in each model of Ra. Thus for every model of Ra there exists an infinite set of individuals. Thus according to Definition 2.1.2.1 we have $Ra \models \exists_{\infty}^{\{faux\}} x\, true$. The third point is evident according to the form of the blocks and the definition of the $(\succ)$-solved block.

Property 2.2.3.8 *The theory Ra is decomposable.*

Proof. We show that Ra satisfies the conditions of Definition 2.2.1.1. The sets A, A', A'', A''' and $\Psi(u)$ are chosen as follows:

- A is the set of blocks.
- A' is the set of formulas of the form $\exists \varepsilon\, \alpha'$ where α' is either a $(\succ)$-solved block or the formula *false*.
- A'' is the set of formulas of the form $\exists \bar{x}''\, true$.
- A''' is the set of formulas of the form $\exists \bar{x}''' \alpha'''$ with α''' a $(\succ)$-solved block and $\bar{x}'''$ the vector of the leaders of the equations of α'''.
- $\Psi(u) = \{false\}$.

Let us denote by BL the set of blocks. It is clear that A', A'' and A''' contain formulas of the form $\exists \bar{x}\, \alpha$ with $\alpha \in BL$. Let us show that BL is Ra-closed. According to the definition of BL we have $BL \subseteq AT$. On the other hand BL is closed for the conjunction. Finally, if α is a flat formula then : if it is the formula *true*, *false*, $x = 0$ or $x = 1$ then it is already a block[7], else, the following rules[8] transform α into a block:

$$\begin{array}{lcl} x = y & \Longrightarrow & x + (-1).y = 0.1 \\ x = -y & \Longrightarrow & x + y = 0.1 \\ x = y + z & \Longrightarrow & x + (-1).y + (-1).z = 0.1 \end{array}$$

[7] Because according to Notation 2.2.3.4 the notations 0.1, 1.1 and $1.x$ represent the terms 0, 1 and x respectively.

[8] These rules are true in Ra and deduced from the axiomatization of Ra

Thus, BL is Ra-closed. Let us show that Ra satisfies the first condition of Definition 2.2.1.1. Let $\alpha \in BL$ and ψ a formula. Let $\bar{x}$ be a vector of variables. Let us choose an order $\succ$ such that the variables of $\bar{x}$ are greater than the free variables of $\exists \bar{x}\, \alpha$. According to Property 2.2.3.6 two cases arise:

Either α is equivalent to *false* in Ra and thus the formula $\exists \bar{x} \alpha \wedge \psi$ is equivalent in Ra to a decomposed formula of the form

$$\exists \varepsilon\, \mathit{false} \wedge (\exists \varepsilon\, \mathit{true} \wedge (\exists \varepsilon\, \mathit{true} \wedge \psi)).$$

Or, α is equivalent in T to a $(\succ)$-solved block β. Then, let X_l be the set of the variables of $\bar{x}$ which are leader in the equations of β. Let X_n be the set of the variables of $\bar{x}$ which are not leader in the equations of β. The formula $\exists \bar{x} \alpha \wedge \psi$ is equivalent in T to a decomposed formula of the form

$$\exists \bar{x}'\, \alpha' \wedge (\exists \bar{x}''\, \alpha'' \wedge (\exists \bar{x}'''\, \alpha''' \wedge \psi)), \tag{2.11}$$

with $\bar{x}' = \varepsilon$. The formula α' contains the conjunction of the equations of β whose leaders do not belong to X_l, i.e. whose leaders are free in $\exists \bar{x} \beta$. The vector $\bar{x}''$ contains the variables of X_n. The formula α'' is the formula *true*. The vector $\bar{x}'''$ contains the variables of X_l. The formula α''' is the conjunction of the equations of β whose leaders belong to X_l. According to our construction it is clear that $\exists \bar{x}' \alpha' \in A'$, $\exists \bar{x}'' \alpha'' \in A''$ and $\exists \bar{x}''' \alpha \in A'''$. Let us show that (2.11) and $\exists \bar{x} \alpha \wedge \psi$ are equivalent in Ra. Let X, X', X'' and X''' be the sets of the variables of the vectors $\bar{x}$, $\bar{x}'$, $\bar{x}''$ and $\bar{x}'''$. If α is equivalent to *false* in Ra then the equivalence of the decomposition is evident. Else β is a $(\succ)$-solved block and thus according to our construction we have: $X = X' \cup X'' \cup X'''$, $X' \cap X'' = \emptyset$, $X' \cap X''' = \emptyset$, $X'' \cap X''' = \emptyset$, $X' = \emptyset$, for all $x''_i \in X''$ we have $x''_i \notin var(\alpha')$ and for all $x'''_i \in X'''$ we have $x'''_i \notin var(\alpha' \wedge \alpha'')$. This is due to the definition of $(\succ)$-solved blocks and the order $\succ$ which has been chosen such that the quantified variables of $\exists \bar{x}\, \alpha$ are greater than the free variables of $\exists \bar{x}\, \alpha$. On the other hand, each equation of β occurs in $\alpha' \wedge \alpha'' \wedge \alpha'''$ and each equation in $\alpha' \wedge \alpha'' \wedge \alpha'''$ occurs in β and thus $Ra \models \beta \leftrightarrow (\alpha' \wedge \alpha'' \wedge \alpha''')$. We have shown that the vectorial quantifications are coherent and the equivalence $Ra \models \beta \leftrightarrow \alpha' \wedge \alpha'' \wedge \alpha'''$ holds. According to Property 2.2.3.6 we have $Ra \models \alpha \leftrightarrow \beta$ and thus, the decomposition keeps the equivalence in Ra. Let us decompose for example

$$\exists xyz\, 2.v + w = 3.1 \wedge v + x = 2.1 \wedge v + x + 2.z = 4.1$$

Let us choose the order $\succ$ such that $x \succ y \succ z \succ v \succ w$. Let us $(\succ)$-solve the block $2.v + w = 3.1 \wedge v + x = 2.1 \wedge v + x + 2.z = 4.1$. Thus the preceding formula is equivalent in Ra to

$$\exists xyz\, 2.v + w = 3.1 \wedge 2.x + (-1).w = 1 \wedge z = 1$$

Finally this formula is equivalent to the following decomposed formula

$$\exists \varepsilon\, 2.v + w = 3.1 \wedge (\exists y\, \mathit{true} \wedge (\exists xz\, 2.x + (-1).w = 1 \wedge z = 1)).$$

The theory Ra satisfies the second condition of Definition 2.2.1.1 according to the third point of Property 2.2.3.7 and using the fact that $\bar{x}' = \varepsilon$. The theory Ra satisfies the third condition of Definition 2.2.1.1 according to the second point of Property 2.2.3.7. The theory Ra satisfies the fourth condition of Definition 2.2.1.1 according to the first point of Property 2.2.3.7. The theory Ra satisfies the last condition of Definition 2.2.1.1 because A' is of the form $\exists \varepsilon\, \alpha'$ where α' is either a $(\succ)$-solved block or the formula *false*. Thus, if α' does not contain free variables then according to the definition of the $(\succ)$-solved blocks α' does not contain formulas of the form $0 = a_0 1$ and thus α' is either the formula *true* or the formula *false*. □

Note that Ra accepts full elimination of quantifiers. In fact Corollary 2.2.2.2 illustrates this result since for all $\exists \bar{x}' \alpha' \in A'$ we have $\bar{x}' = \varepsilon$.

2.3 A decision procedure in infinite-decomposable theories

Let T be a decomposable theory together with its set of function symbols F and its set of relation symbols R. The sets $\Psi(u)$, A, A', A'' and A''' are now known and fixed.

2.3.1 Normalized formula

Definition 2.3.1.1 *A normalized formula φ of depth $d \geq 1$ is a formula of the form*

$$\neg(\exists \bar{x}\, \alpha \wedge \bigwedge_{i \in I} \varphi_i), \tag{2.12}$$

with I a finite (possibly empty) set, $\alpha \in FL$ and the φ_i's are normalized formulas of depth d_i with $d = 1 + \max\{0, d_1, ..., d_n\}$ and all the quantified variables of φ have distinct names and different from the names of the free variables.

Example 2.3.1.2 *Let f and g two 1-ary function symbol which belong to F. The formula*

$$\neg\left[\exists\varepsilon\, true \wedge \left[\begin{array}{l} \neg(\exists x\, y = fx \wedge x = y \wedge \neg(\exists\varepsilon\, y = gx))\wedge \\ \neg(\exists\varepsilon\, x = z) \end{array}\right]\right]$$

is a normalized formula of depth equals to three. The formulas $\neg(\exists\varepsilon\, true)$ and $\neg(\exists\varepsilon\, false)$ are two normalized formulas of depth 1. The smallest value of a depth of a normalized formula is 1. Normalized formulas of depth 0 are not defined and do not exist.

Property 2.3.1.3 *Every formula φ is equivalent in T to a wnfv normalized formula of depth $d \geq 1$.*

Proof. It is easy to transform any formula into a wnfv normalized formula, it is enough for example to follow the followings steps:

1. Introduce a supplement of equations and existentially quantified variables to transform the conjunctions of equations and relations into conjunctions of flat formulas.
2. Express all the quantifiers, constants and logical connectors with $\neg$, $\wedge$ and $\exists$, using the following transformations[9] of sub-formulas :

$$\begin{array}{lll} (\varphi \vee \phi) & \Longrightarrow & \neg(\neg\varphi \wedge \neg\phi), \\ (\varphi \rightarrow \phi) & \Longrightarrow & \neg(\varphi \wedge \neg\phi), \\ (\varphi \leftrightarrow \phi) & \Longrightarrow & (\neg(\varphi \wedge \neg\phi) \wedge \neg(\phi \wedge \neg\varphi)), \\ (\forall x\, \varphi) & \Longrightarrow & \neg(\exists x\, \neg\varphi). \end{array}$$

3. If the formula φ obtained does not start with the logical symbol $\neg$, then replace it by $\neg(\exists\varepsilon\, true \wedge \neg\varphi)$.
4. Name the quantified variables by distinct names and different from the names of the free variables.
5. Lift the quantifier before the conjunction, i.e. $\varphi \wedge (\exists \bar{x}\, \psi)$ or $(\exists \bar{x}\, \psi) \wedge \varphi$, becomes $\exists \bar{x}\, \varphi \wedge \psi$ because the free variables of φ are distinct from those of $\bar{x}$.

[9] These equivalences are true in the empty theory and thus in any theory T.

6. Group quantified variables into a vectorial quantifier, i.e. $\exists\bar{x}(\exists\bar{y}\,\varphi)$ or $\exists\bar{x}\exists\bar{y}\,\varphi$ becomes $\exists\overline{xy}\,\varphi$.

7. Insert empty vectors and formulas of the form *true* to get the normalized form using the following transformations of sub-formulas:

$$\neg(\bigwedge_{i\in I}\neg\varphi_i) \Longrightarrow \neg(\exists\varepsilon\, true \wedge \bigwedge_{i\in I}\neg\varphi_i), \tag{2.13}$$

$$\neg(\alpha \wedge \bigwedge_{i\in I}\neg\varphi_i) \Longrightarrow \neg(\exists\varepsilon\, \alpha \wedge \bigwedge_{i\in I}\neg\varphi_i), \tag{2.14}$$

$$\neg(\exists\bar{x} \bigwedge_{j\in J}\neg\varphi_j) \Longrightarrow \neg(\exists\bar{x}\, true \wedge \bigwedge_{j\in J}\neg\varphi_j). \tag{2.15}$$

with $\alpha \in FL$, I a finite (possibly empty) set and J a finite non-empty set.

If the starting formula does not contain the logical symbol $\leftrightarrow$ then this transformation will be linear, i.e. there exists a constant k such that $n_2 \leq kn_1$, where n_1 is the size of the starting formula and n_2 the size of the normalized formula. We show easily by contradiction that the final formula obtained after application of these steps is normalized. □

Example 2.3.1.4 *Let f be a 2-ary function symbols which belong to F. Let us apply the preceding steps to transform the following formula into a normalized formula which is equivalent in T:*

$$(fuv = fwu \wedge (\exists x\, u = x)) \vee (\exists u \forall w\, u = fvw).$$

Note that the formula does not start with $\neg$ and the variables u and w are free in $fuv = fwu \wedge (\exists x\, u = x)$ and bound in $\exists u \forall w\, u = fvw$.
Step 1: Let us first transform the equations into flat equations. The preceding formula is equivalent in T to

$$(\exists u_1\, u_1 = fuv \wedge u_1 = fwu \wedge (\exists x\, u = x)) \vee (\exists u \forall w\, u = fvw). \tag{2.16}$$

Step 2: Let us now express the quantifier $\forall$ with $\neg$, $\wedge$ and $\exists$. Thus, the formula (2.16) is equivalent in T to

$$(\exists u_1\, u_1 = fuv \wedge u_1 = fwu \wedge (\exists x\, u = x)) \vee (\exists u\, \neg(\exists w\, \neg(u = fvw))).$$

Let us also express the logical symbol $\vee$ with $\neg$, $\wedge$ and $\exists$. The preceding formula is equivalent in T to

$$\neg(\neg(\exists u_1\, u_1 = fuv \wedge u_1 = fwu \wedge (\exists x\, u = x)) \wedge \neg(\exists u\, \neg(\exists w\, \neg(u = fvw)))). \tag{2.17}$$

Step 3: The formula starts with $\neg$, then we move to Step 4.
Step 4: The quantified variables u and w in $(\exists u\, \neg(\exists w\, \neg(u = fvw)))$ must be renamed. Thus, the formula (2.17) is equivalent in T to

$$\neg(\neg(\exists u_1\, u_1 = fuv \wedge u_1 = fwu \wedge (\exists x\, u = x)) \wedge \neg(\exists u_2\, \neg(\exists w_1\, \neg(u_2 = fvw_1)))).$$

Step 5: By lifting the existential quantifier $\exists x$, the preceding formula is equivalent in T to

$$\neg(\neg(\exists u_1\, \exists x\, u_1 = fuv \wedge u_1 = fwu \wedge u = x) \wedge \neg(\exists u_2\, \neg(\exists w_1\, \neg(u_2 = fvw_1)))).$$

Step 6: Let us group the two quantified variables x and u_1 into a vectorial quantifier. Thus, the preceding formula is equivalent in T to

$$\neg(\neg(\exists u_1 x\, u_1 = fuv \wedge u_1 = fwu \wedge u = x) \wedge \neg(\exists u_2 \neg(\exists w_1 \neg(u_2 = fvw_1)))).$$

Step 7: Let us introduces empty vectors of variables and formulas of the form true to get the normalized formula. According to the rule (2.13), the preceding formula is equivalent in T to

$$\neg\left[\exists \varepsilon\, true \wedge \begin{bmatrix} \neg(\exists u_1 x\, u_1 = fuv \wedge u_1 = fwu \wedge u = x)\wedge \\ \neg(\exists u_2 \neg(\exists w_1 \neg(u_2 = fvw_1))) \end{bmatrix}\right],$$

then using the rule (2.14) it is equivalent to

$$\neg\left[\exists \varepsilon\, true \wedge \begin{bmatrix} \neg(\exists u_1 x\, u_1 = fuv \wedge u_1 = fwu \wedge u = x)\wedge \\ \neg(\exists u_2 \neg(\exists w_1 \neg(\exists \varepsilon\, u_2 = fvw_1))) \end{bmatrix}\right],$$

and finally using the rule (2.15) it is equivalent to

$$\neg\left[\exists \varepsilon\, true \wedge \begin{bmatrix} \neg(\exists u_1 x\, u_1 = fuv \wedge u_1 = fwu \wedge u = x)\wedge \\ \neg(\exists u_2\, true \wedge \neg(\exists w_1\, true \wedge \neg(\exists \varepsilon\, u_2 = fvw_1))) \end{bmatrix}\right],$$

This is a normalized formula of depth 4.

2.3.2 Working formula

Definition 2.3.2.1 *A* working formula φ *of depth $d \geq 1$ is a formula of the form*

$$\neg(\exists \bar{x}\, \alpha \wedge \bigwedge_{i \in I} \varphi_i), \tag{2.18}$$

with I a finite (possibly empty) set, $\alpha \in A$ and the φ_i's are working formulas of depth d_i with $d = 1 + \max\{0, d_1, ..., d_n\}$ and all the quantified variables of φ have distinct names and different from the names of the free variables.

Property 2.3.2.2 *Every formula is equivalent in T to a wnfv working formula.*

Proof. Let φ be a formula. According to Property 2.3.1.3 φ is equivalent in T to a wnfv normalized formula ϕ of the form

$$\neg(\exists \bar{x}\, \alpha \wedge \bigwedge_{i \in I} \varphi_i), \tag{2.19}$$

with $\alpha \in FL$ and all the φ_i are normalized formulas. Since T is decomposable then according to Definition 2.2.1.1 the set A is T-closed, i.e. (i) $A \subseteq AT$, (ii) A is closed for the conjunction and (iii) every flat formula is equivalent in T to a formula which belongs to A. Since $\alpha \in FL$, then according to (iii) α is equivalent in T to a conjunction β of elements of A. According to (ii) β belongs to A. Thus, the formula (2.19) is equivalent in T to

$$\neg(\exists \bar{x}\, \beta \wedge \bigwedge_{i \in I} \varphi_i), \tag{2.20}$$

with $\beta \in A$. By repeating the preceding steps recursively on each sub-normalized formula φ_i of (2.20) we get a working formula. □

Example 2.3.2.3 *In the theory Ra of additive rational numbers, the formula*

$$\neg\left[\exists\varepsilon\, true \wedge \left[\begin{array}{l}\neg(\exists x\, y = -z \wedge z = x + y)\wedge\\ \neg(\exists\, true \wedge \neg(\exists w\, true \wedge \neg(\exists\varepsilon\, z = w)))\end{array}\right]\right],$$

is a normalized formula of depth 4 which is equivalent in Ra to the following working formula

$$\neg\left[\exists\varepsilon\, true \wedge \left[\begin{array}{l}\neg(\exists x\, y + z = 0.1 \wedge z + (-1).x + (-1).y = 0.1)\wedge\\ \neg(\exists\, true \wedge \neg(\exists w\, true \wedge \neg(\exists\varepsilon\, z + (-1).w = 0.1)))\end{array}\right]\right].$$

The formula $\neg(\exists\varepsilon\, z + (-1).w = 0.1)$ *is a sub-working formula.*

Definition 2.3.2.4 *A* solved *formula is a working formula of the form*

$$\neg(\exists\bar{x}'\,\alpha' \wedge \bigwedge_{i\in I}\neg(\exists\bar{y}'_i\,\beta'_i)), \tag{2.21}$$

with I *a finite (possibly empty) set,* $\exists\bar{x}'\alpha' \in A'$*,* $\exists\bar{y}'_i\beta'_i \in A'$ *for all* $i \in I$*,* α' *different from the formula false and all the* β'_i *are different from the formulas true and false.*

Property 2.3.2.5 *Let* φ *be a conjunction of solved formulas without free variables. The conjunction* φ *is either the formula* $\neg true$ *or the formula true.*

Proof. Recall first that we write $\bigwedge_{i\in I}\varphi_i$, and call *conjunction* each formula of the form $\varphi_{i_1} \wedge \varphi_{i_2} \wedge ... \wedge \varphi_{i_n} \wedge true$. Let φ be a conjunction of solved formulas without free variables. According to Definition 2.3.2.4, φ is of the form

$$(\bigwedge_{i\in I}\neg(\exists\bar{x}'_i\alpha'_i \wedge \bigwedge_{j\in J_i}\neg(\exists\bar{y}'_{ij}\beta'_{ij}))) \wedge true \tag{2.22}$$

with

1. I a finite (possibly empty) set,
2. $(\exists\bar{x}'_i\alpha'_i) \in A'$ for all $i \in I$,
3. $(\exists\bar{y}'_{ij}\beta'_{ij}) \in A'$ for all $i \in I$ and $j \in J_i$,
4. α'_i different from *false* for all $i \in I$,
5. β'_{ij} different from *true* and *false* for all $i \in I$ and $j \in J_i$.

Since these solved formulas don't have free variables and since T is a decomposable theory then according to the fifth point of Definition 2.2.1.1 of a decomposable theory and the conditions 2 and 3 of (2.22) we have:
(*) each formula $\exists\bar{x}'_i\alpha'_i$ and each formula $\exists\bar{y}'_{ij}\beta'_{ij}$ is either the formula $\exists\varepsilon true$ or $\exists\varepsilon false$.
According to (*) and the condition 5 of (2.22), all the sets J_i must be empty, thus φ is of the form

$$(\bigwedge_{i\in I}\neg(\exists\bar{x}'_i\alpha'_i)) \wedge true \tag{2.23}$$

According to (*) and (2.23), the formula φ is of the form

$$(\bigwedge_{i\in I'}\neg(\exists\varepsilon false)) \wedge (\bigwedge_{j\in I-I'}\neg(\exists\varepsilon true)) \wedge true$$

According to the condition 4 of (2.22), the set I' must be empty and thus φ is of the form

$$(\bigwedge_{i\in I} \neg(\exists\varepsilon\, true)) \wedge true$$

If $I = \emptyset$ then φ is the formula *true*, else, according to our assumptions, we do not distinguish two formulas which can be made equal using the following transformation of the sub-formulas:

$$\varphi \wedge \varphi \Longrightarrow \varphi,\ \ \varphi \wedge \psi \Longrightarrow \psi \wedge \varphi,\ \ (\varphi \wedge \psi) \wedge \phi \Longrightarrow \varphi \wedge (\psi \wedge \phi),$$
$$\varphi \wedge true \Longrightarrow \varphi,\ \ \varphi \vee false \Longrightarrow \varphi.$$

Thus φ is the formula

$$\neg true$$

□

Property 2.3.2.6 *Every solved formula is equivalent in T to a wnfv Boolean combination of elements of A'.*

Proof. Let φ be a solved formula. According to Definition 2.3.2.4, the formula φ is of the form

$$\neg(\exists \bar{x}'\, \alpha' \wedge \bigwedge_{i\in I} \neg(\exists \bar{y}_i'\, \beta_i')),$$

with $\exists\bar{x}'\alpha' \in A'$ and $\exists\bar{y}_i'\beta_i' \in A'$ for all $i \in I$. Since $\exists\bar{x}'\alpha' \in A'$ then according to Definition 2.2.1.1 $T \models \exists?\bar{x}'\alpha'$ and thus according to Corollary 2.1.1.4, φ is equivalent in T to the following wnfv formula

$$\neg((\exists \bar{x}'\, \alpha') \wedge \bigwedge_{i\in I} \neg(\exists \bar{x}'\, \alpha' \wedge (\exists \bar{y}_i'\, \beta_i'))).$$

According to the definition of working formulas all the quantified variables of φ have distinct names and different from the names of the free variables, thus the preceding formula is equivalent in T to the wnfv formula

$$\neg((\exists \bar{x}'\, \alpha') \wedge \bigwedge_{i\in I} \neg(\exists \bar{x}'\bar{y}_i'\, \alpha' \wedge \beta_i')).$$

Since $\exists\bar{x}'\alpha' \in A'$ and $\exists\bar{y}_i'\beta_i' \in A'$ for all $i \in I$, then $\alpha' \in A$ and $\beta_i' \in A$. Since A is T-closed then it closed for the conjunction and thus $\alpha' \wedge \beta_i' \in A$ for all $i \in I$. According to Property 2.2.1.2 the preceding formula is equivalent in T to a wnfv formula of the form

$$\neg((\exists \bar{x}'\, \alpha') \wedge \bigwedge_{i\in I} \neg(\exists \bar{z}_i'\, \delta_i')),$$

with $\exists\bar{x}'\alpha' \in A'$ and $\exists\bar{z}_i'\delta_i' \in A'$ for all $i \in I$. Which is finally equivalent in T to

$$(\neg(\exists \bar{x}'\, \alpha')) \vee \bigvee_{i\in I} (\exists \bar{z}_i'\, \delta_i').$$

□

2.3.3 The rewriting rules

We present now the rewriting rules which transform a working formula φ of any depth d into a wnfv conjunction ϕ of solved formulas which is equivalent to φ in T. To apply the rule $p_1 \Longrightarrow p_2$ to the working formula p means to replace in p, a sub-formula p_1 by the formula p_2, by considering that the connector $\wedge$ is associative and commutative.

$$(1)\quad \neg\begin{bmatrix} \exists \bar{x}\, \alpha \wedge \varphi \wedge \\ \neg(\exists \bar{y}\, \mathit{true}) \end{bmatrix} \quad\Longrightarrow\quad \mathit{true}$$

$$(2)\quad \neg\begin{bmatrix} \exists \bar{x}\, \mathit{false} \wedge \varphi \end{bmatrix} \quad\Longrightarrow\quad \mathit{true}$$

$$(3)\quad \neg\begin{bmatrix} \exists \bar{x}\, \alpha \wedge \\ \bigwedge_{i\in I} \neg(\exists \bar{y}_i\, \beta_i) \end{bmatrix} \quad\Longrightarrow\quad \neg\begin{bmatrix} \exists \bar{x}'\bar{x}''\, \alpha' \wedge \alpha'' \wedge \\ \bigwedge_{i\in I} \neg(\exists \bar{x}'''\bar{y}_i\, \alpha''' \wedge \beta_i)^* \end{bmatrix}$$

$$(4)\quad \neg\begin{bmatrix} \exists \bar{x}\, \alpha \wedge \\ \bigwedge_{i\in I} \neg(\exists \bar{y}'_i\, \beta'_i) \end{bmatrix} \quad\Longrightarrow\quad \neg\begin{bmatrix} \exists \bar{x}'\, \alpha' \wedge \\ \bigwedge_{i\in I'} \neg(\exists \bar{y}'_i\, \beta'_i) \end{bmatrix}$$

$$(5)\quad \neg\begin{bmatrix} \exists \bar{x}\, \alpha \wedge \varphi \wedge \\ \neg\begin{bmatrix} \exists \bar{y}'\, \beta' \wedge \\ \bigwedge_{i\in I} \neg(\exists \bar{z}'_i\, \delta'_i) \end{bmatrix} \end{bmatrix} \quad\Longrightarrow\quad \begin{bmatrix} \neg(\exists \bar{x}\, \alpha \wedge \varphi \wedge \neg(\exists \bar{y}'\, \beta')) \wedge \\ \bigwedge_{i\in I} \neg(\exists \bar{x}\bar{y}'\bar{z}'_i\, \alpha \wedge \beta' \wedge \delta'_i \wedge \varphi)^* \end{bmatrix}$$

with $\alpha \in A$, φ a conjunction of working formulas and I a finite (possibly empty) set. In the rule (3), the formula $\exists \bar{x}\, \alpha$ is equivalent in T to a decomposed formula of the form $\exists \bar{x}'\, \alpha' \wedge (\exists \bar{x}''\, \alpha'' \wedge (\exists \bar{x}'''\, \alpha'''))$ with $\exists \bar{x}'\, \alpha' \in A'$, $\exists \bar{x}''\, \alpha'' \in A''$, $\exists \bar{x}'''\, \alpha''' \in A'''$ and $\exists \bar{x}'''\, \alpha'''$ different from $\exists \varepsilon\, \mathit{true}$. All the β_i's belong to A. The formula $(\exists \bar{x}'''\bar{y}_i\, \alpha''' \wedge \beta_i)^*$ is the formula $(\exists \bar{x}'''\bar{y}_i\, \alpha''' \wedge \beta_i)$ in which we have renamed the variables of $\bar{x}'''$ by distinct names and different from the names of the free variables. In the rule (4), the formula $\exists \bar{x}\, \alpha$ is not an element of A' and is equivalent in T to a decomposed formula of the form $\exists \bar{x}'\, \alpha' \wedge (\exists \bar{x}''\, \alpha'' \wedge (\exists \varepsilon\, \mathit{true}))$ with $\exists \bar{x}'\, \alpha' \in A'$ and $\exists \bar{x}''\, \alpha'' \in A''$. Each formula $\exists \bar{y}'_i\, \beta'_i$ is an element of A'. I' is the set of the $i \in I$ such that $\exists \bar{y}'_i \beta'_i$ does not have free occurrences of any variables of $\bar{x}''$. In the rule (5), $I \neq \emptyset$, $\exists \bar{y}'\, \beta' \in A'$ and $\exists \bar{z}'_i\, \delta'_i \in A'$ for all $i \in I$. The formula $(\exists \bar{x}\bar{y}'\bar{z}'_i\, \alpha \wedge \beta' \wedge \delta'_i \wedge \varphi)^*$ is the formula $(\exists \bar{x}\bar{y}'\bar{z}'_i\, \alpha \wedge \beta' \wedge \delta'_i \wedge \varphi)$ in which we have renamed the variables of $\bar{x}$ and $\bar{y}'$ by distinct names and different from the names of the free variables.

Property 2.3.3.1 *Every repeated application of the preceding rewriting rules on any working formula φ, terminates and produces a wnfv conjunction ϕ of solved formulas which is equivalent to φ in T.*

Proof, first part: The application of the rewriting rules terminates. Let us consider the 3-tuples (n_1, n_2, n_3) where the n_i's are the following positive integers:

- $n_1 = \alpha(p)$, where the function α is defined as follows:
  - $\alpha(\mathit{true}) = 0$,

- $\alpha(\neg(\exists\bar{x}\, a \wedge \varphi)) = 2^{\alpha(\varphi)}$,
- $\alpha(\bigwedge_{i\in I}\varphi_i) = \sum_{i\in I}\alpha(\varphi_i)$,

with $a \in A$, φ a conjunction of working formulas and the φ_i's working formulas. Note that if $\alpha(p_2) < \alpha(p_1)$ then $\alpha(p[p_2]) < \alpha(p)$ where $p[p_2]$ is the formula obtained from p when we replace the occurrence of the formula p_1 in p by p_2. This function has been introduced in [41] and [7] to show the non-elementary complexity of every algorithm solving propositions in the theory of finite or infinite trees. It has also the property to decrease if the depth of the working formula decreases after application of distribution as it is done in our rule (5).

- $n_2 = \beta(p)$, where the function β is defined as follows:
  - $\beta(true) = 0$,
  - $\beta(\neg(\exists\bar{x}\, a \wedge \bigwedge_{i\in I}\varphi_i)) = \left\{ \begin{array}{l} 4^{1+\sum_{i\in I}\beta(\varphi_i)} \text{ if } \exists\bar{x}'''\alpha''' \neq \exists\varepsilon true, \\ 1+\sum_{i\in I}\beta(\varphi_i) \text{ if } \exists\bar{x}'''\alpha''' = \exists\varepsilon true \end{array} \right\}$
  with the φ_i's working formulas and $T \models (\exists\bar{x}\alpha) \leftrightarrow (\exists\bar{x}'\alpha' \wedge (\exists\bar{x}''\alpha'' \wedge (\exists\bar{x}'''\alpha''')))$.

  We show that:

$$\beta(\neg(\exists\bar{x}\,\alpha \wedge \bigwedge_{i\in I}\neg(\exists\bar{y}_i\,\lambda_i))) > \beta(\neg(\exists\bar{z}\delta \wedge \bigwedge_{i\in I}\neg(\exists w_i\,\gamma_i)))$$

  where the formula $\exists\bar{x}\,\alpha$ is equivalent in T to a decomposed formula of the form $\exists\bar{x}'\alpha' \wedge (\exists\bar{x}''\alpha'' \wedge (\exists\bar{x}'''\,\alpha'''))$ with $\exists\bar{x}'''\,\alpha''' \neq \exists\varepsilon\, true$, the formula $\exists\bar{z}\,\delta$ is equivalent in T to a decomposed formula of the form $\exists\bar{z}'\,\delta' \wedge (\exists\bar{z}''\,\delta'' \wedge (\exists\varepsilon\, true))$ and all the λ_i and γ_i belong to A and have no particular conditions.

- n_3 is the number of the sub-formulas of the form $\neg(\exists\bar{x}\alpha \wedge \varphi)$ with $\exists\bar{x}\alpha \notin A'$ and φ a conjunction of working formulas.

For each rule, there exists a row i such that the application of this rule decreases or does not change the values of the n_j's, with $1 \leq j < i$, and decreases the value of n_i. The row i is equal to: 1 for the rules (1), (2) and (5), 2 for the rule (3) and 3 for the rule (4). To each sequence of formulas obtained by a finite application of the preceding rewriting rules, we can associate a series of 3-tuples (n_1, n_2, n_3) which is strictly decreasing in the lexicographic order. Since the n_i's are positive integers, they cannot be negative, thus, this series of 3-tuples is a finite series and the application of the rewriting rules terminates.

Proof, second part: Let us show now that for each rule of the form $p \Longrightarrow p'$ we have $T \models p \leftrightarrow p'$ and the formula p' remains a conjunction of working formulas. It is clear that the rules (1) and (2) are correct.

Correctness of the rule (3):

$$\neg\left[\begin{array}{l} \exists\bar{x}\,\alpha\wedge \\ \bigwedge_{i\in I}\neg(\exists\bar{y}_i\,\beta_i) \end{array}\right] \Longrightarrow \neg\left[\begin{array}{l} \exists\bar{x}'\bar{x}''\,\alpha' \wedge \alpha''\wedge \\ \bigwedge_{i\in I}\neg(\exists\bar{x}'''\bar{y}_i\,\alpha''' \wedge \beta_i) \end{array}\right]$$

where the formula $\exists \bar{x}\,\alpha$ is equivalent in T to a decomposed formula of the form $\exists \bar{x}'\,\alpha' \wedge (\exists \bar{x}''\,\alpha'' \wedge (\exists \bar{x}'''\,\alpha'''))$ with $\exists \bar{x}'\,\alpha' \in A'$, $\exists \bar{x}''\,\alpha'' \in A''$, $\exists \bar{x}'''\,\alpha''' \in A'''$ and $\exists \bar{x}'''\,\alpha'''$ different from $\exists \varepsilon\, true$.

Let us show the correctness of this rule. According to the conditions of application of this rule, the formula $\exists \bar{x}\,\alpha$ is equivalent in T to a decomposed formula of the form $\exists \bar{x}'\,\alpha' \wedge (\exists \bar{x}''\,\alpha'' \wedge (\exists \bar{x}'''\,\alpha'''))$ with $\exists \bar{x}'\,\alpha' \in A'$, $\exists \bar{x}''\,\alpha'' \in A''$, $\exists \bar{x}'''\,\alpha''' \in A'''$ and $\exists \bar{x}'''\,\alpha'''$ different from $\exists \varepsilon\, true$. Thus, the left formula of this rewriting rule is equivalent in T to the formula

$$\neg(\exists \bar{x}'\,\alpha' \wedge (\exists \bar{x}''\alpha'' \wedge (\exists \bar{x}'''\alpha''' \wedge \bigwedge_{i\in I} \neg(\exists \bar{y}_i\,\beta_i)))).$$

Since $\exists \bar{x}'''\,\alpha''' \in A'''$, then according to the fourth point of Definition 2.2.1.1 we have $T \models \exists! \bar{x}'''\alpha'''$, thus using Corollary 2.1.1.7 the preceding formula is equivalent in T to

$$\neg(\exists \bar{x}'\,\alpha' \wedge (\exists \bar{x}''\alpha'' \wedge \bigwedge_{i\in I} \neg(\exists \bar{x}'''\alpha''' \wedge (\exists \bar{y}_i\,\beta_i))))$$

According to the definition of the working formula the quantified variables have distinct names and different from the names of the free variables, thus, we can lift the quantifications and then the preceding formula is equivalent in T to

$$\neg(\exists \bar{x}'\,\alpha' \wedge (\exists \bar{x}''\alpha'' \wedge \bigwedge_{i\in I} \neg(\exists \bar{x}'''\bar{y}_i\,\alpha''' \wedge \beta_i)))$$

i.e. to

$$\neg(\exists \bar{x}'\bar{x}''\,\alpha' \wedge \alpha'' \wedge \bigwedge_{i\in I} \neg(\exists \bar{x}'''\bar{y}_i\,\alpha''' \wedge \beta_i)^*),$$

where the formula $(\exists \bar{x}'''\bar{y}_i\,\alpha''' \wedge \beta_i)^*$ is the formula $(\exists \bar{x}'''\bar{y}_i\,\alpha''' \wedge \beta_i)$ in which we have renamed the variables of $\bar{x}'''$ by distinct names and different from the names of the free variables. Thus, the rewriting rule (3) is correct in T.

Correctness of the rule (4):

$$\neg\begin{bmatrix} \exists \bar{x}\,\alpha \wedge \\ \bigwedge_{i\in I} \neg(\exists \bar{y}_i'\,\beta_i') \end{bmatrix} \Longrightarrow \neg\begin{bmatrix} \exists \bar{x}'\,\alpha' \wedge \\ \bigwedge_{i\in I'} \neg(\exists \bar{y}_i'\,\beta_i') \end{bmatrix}$$

where the formula $\exists \bar{x}\,\alpha$ is not an element of A' and is equivalent in T to a decomposed formula of the form $\exists \bar{x}'\,\alpha' \wedge (\exists \bar{x}''\,\alpha'' \wedge (\exists \varepsilon\, true))$ with $\exists \bar{x}'\,\alpha' \in A'$ and $\exists \bar{x}''\,\alpha'' \in A''$. Each formula $\exists \bar{y}_i'\,\beta_i'$ is an element of A'. I' is the set of the $i \in I$ such that $\exists \bar{y}_i'\beta_i'$ does not have free occurrences of any variables of $\bar{x}''$.

Let us show the correctness of this rule. According to the conditions of application of this rule, the formula $\exists \bar{x}\,\alpha$ is equivalent in T to a decomposed formula of the form $\exists \bar{x}'\,\alpha' \wedge (\exists \bar{x}''\,\alpha'' \wedge (\exists \varepsilon\, true))$ with $\exists \bar{x}'\,\alpha' \in A'$ and $\exists \bar{x}''\,\alpha'' \in A''$. Moreover, each formula $\exists \bar{y}_i'\,\beta_i'$ belongs to A'. Thus, the left formula of this rewriting rule is equivalent in T to the formula

$$\neg(\exists \bar{x}'\,\alpha' \wedge (\exists \bar{x}''\alpha'' \wedge \bigwedge_{i\in I} \neg(\exists \bar{y}_i'\,\beta_i')))$$

Let us denote by I_1, the set of the $i \in I$ such that x_n'' does not have free occurrences in the formula $\exists \bar{y}_i'\beta_i'$, thus, the preceding formula is equivalent in T to

$$\neg(\exists \bar{x}'\alpha' \wedge (\exists x_1''...\exists x_{n-1}'' \begin{bmatrix} (\bigwedge_{i\in I_1} \neg(\exists \bar{y}_i'\beta_i')) \wedge \\ (\exists x_n''\,\alpha'' \wedge \bigwedge_{i\in I-I_1} \neg(\exists \bar{y}_i'\beta_i')) \end{bmatrix})). \qquad (2.24)$$

Since $\exists \bar{x}''\alpha'' \in A''$ and $\exists \bar{y}'_i\beta'_i \in A'$ for every $i \in I - I_1$, then according to Property 2.1.2.2 and the conditions 2 and 3 of Definition 2.2.1.1, the formula (2.24) is equivalent in T to

$$\neg(\exists \bar{x}'\alpha' \wedge (\exists x''_1 ... \exists x''_{n-1} \, (\mathit{true} \wedge \textstyle\bigwedge_{i\in I_1} \neg(\exists \bar{y}'_i\beta'_i)))). \tag{2.25}$$

By repeating the three preceding steps $(n-1)$ times, by denoting by I_k the set of the $i \in I_{k-1}$ such that $x''_{(n-k+1)}$ does not have free occurrences in $\exists \bar{y}'_i\beta'_i$, and by using $(n-1)$ times Property 2.1.2.3, the preceding formula is equivalent in T to

$$\neg(\exists \bar{x}'\alpha' \wedge \textstyle\bigwedge_{i\in I_n} \neg(\exists \bar{y}'_i\beta'_i)),$$

Thus, the rule (4) is correct in T.

Correctness of the rule (5):

$$\neg \left[\begin{array}{l} \exists \bar{x}\, \alpha \wedge \varphi \wedge \\ \neg \left[\begin{array}{l} \exists \bar{y}'\, \beta' \wedge \\ \bigwedge_{i\in I} \neg(\exists \bar{z}'_i\, \delta'_i) \end{array} \right] \end{array} \right] \Longrightarrow \left[\begin{array}{l} \neg(\exists \bar{x}\, \alpha \wedge \varphi \wedge \neg(\exists \bar{y}'\, \beta')) \wedge \\ \bigwedge_{i\in I} \neg(\exists \bar{x}\bar{y}'\bar{z}'_i\, \alpha \wedge \beta' \wedge \delta'_i \wedge \varphi)^* \end{array} \right]$$

where $I \neq \emptyset$ and the formulas $\exists \bar{y}'\, \beta'$ and $\exists \bar{z}'_i\, \delta'_i$ are elements of A' for all $i \in I$.

Let us show the correctness of this rule. Since $\exists \bar{y}'\beta' \in A'$ then according to the second point of Definition 2.2.1.1 we have $T \models \exists ? \bar{y}'\beta'$, thus, using Corollary 2.1.1.4 the preceding formula is equivalent to

$$\neg \left[\begin{array}{l} \exists \bar{x}\, \alpha \wedge \varphi \wedge \\ \neg \left[\; (\exists \bar{y}'\, \beta') \wedge \bigwedge_{i\in I} \neg(\exists \bar{y}'\, \beta' \wedge (\exists \bar{z}'_i\, \delta'_i)) \; \right] \end{array} \right]$$

According to the definition of the working formula the quantified variables have distinct names and different from the names of the free variables, thus we can lift the quantifications and then the preceding formula is equivalent in T to

$$\neg \left[\begin{array}{l} \exists \bar{x}\, \alpha \wedge \varphi \wedge \\ \neg \left[\; (\exists \bar{y}'\, \beta') \wedge \bigwedge_{i\in I} \neg(\exists \bar{y}'\bar{z}'_i\, \beta' \wedge \delta'_i) \; \right] \end{array} \right]$$

thus to

$$\neg \left[\begin{array}{l} \exists \bar{x}\, \alpha \wedge \varphi \wedge \\ \left[\; (\neg(\exists \bar{y}'\, \beta')) \vee \bigvee_{i\in I} (\exists \bar{y}'\bar{z}'_i\, \beta' \wedge \delta'_i) \; \right] \end{array} \right]$$

After having distributed the $\wedge$ on the $\vee$ and lifted the quantification $\exists \bar{y}'\bar{z}'_i$ we get

$$\neg \left[\begin{array}{l} (\exists \bar{x}\, \alpha \wedge \varphi \wedge \neg(\exists \bar{y}'\, \beta')) \vee \\ \bigvee_{i\in I} (\exists \bar{x}\bar{y}'\bar{z}'_i\, \alpha \wedge \varphi \wedge \beta' \wedge \delta'_i) \end{array} \right]$$

which is equivalent in T to

$$\left[\begin{array}{l} \neg(\exists \bar{x}\, \alpha \wedge \varphi \wedge \neg(\exists \bar{y}'\, \beta')) \wedge \\ \bigwedge_{i\in I} \neg(\exists \bar{x}\bar{y}'\bar{z}'_i\, \alpha \wedge \varphi \wedge \beta' \wedge \delta'_i) \end{array} \right] \tag{2.26}$$

In order to satisfy the definition of the working formulas we must rename the variables of $\bar{x}$ and $\bar{y}'$ by distinct names and different from the names of the free variables. Let us denote by

$(\exists \bar{x}\bar{y}'\bar{z}'_i\, \alpha \wedge \varphi \wedge \beta' \wedge \delta'_i)^*$ the formula $(\exists \bar{x}\bar{y}'\bar{z}'_i\, \alpha \wedge \varphi \wedge \beta' \wedge \delta'_i)$ in which we have renamed the variables of $\bar{x}$ and $\bar{y}'$. Thus, the formula (2.26) is equivalent in T to

$$\left[\begin{array}{l} \neg(\exists \bar{x}\, \alpha \wedge \varphi \wedge \neg(\exists \bar{y}'\, \beta')) \wedge \\ \bigwedge_{i \in I} \neg(\exists \bar{x}\bar{y}'\bar{z}'_i\, \alpha \wedge \varphi \wedge \beta' \wedge \delta'_i)^* \end{array}\right]$$

Thus, the rule (5) is correct in T. It is very important to note that all the rewriting rules preserve the equivalence of the formulas without adding new free variables.

Proof, third part: Every finite application of the rewriting rules on a working formula produces a wnfv conjunction of solved formulas.

Recall that we write $\bigwedge_{i \in I} \varphi_i$, and call *conjunction* each formula of the form $\varphi_{i_1} \wedge \varphi_{i_2} \wedge ... \wedge \varphi_{i_n} \wedge true$. In particular, for $I = \emptyset$, the conjunction $\bigwedge_{i \in I} \varphi_i$ is reduced to *true*. Moreover, we do not distinguish two formulas which can be made equal using the following transformation of the sub-formulas:

$$\varphi \wedge \varphi \Longrightarrow \varphi, \quad \varphi \wedge \psi \Longrightarrow \psi \wedge \varphi, \quad (\varphi \wedge \psi) \wedge \phi \Longrightarrow \varphi \wedge (\psi \wedge \phi),$$
$$\varphi \wedge true \Longrightarrow \varphi, \quad \varphi \vee false \Longrightarrow \varphi.$$

Let us show first that every substitution of a sub-working formula of a conjunction of working formulas by a conjunction of working formulas produces a conjunction of working formulas. Let $\bigwedge_{i \in I} \varphi_i$ be a conjunction of working formulas. Let φ_k with $k \in I$ an element of this conjunction of depth d_k. Two cases arise:

1. Either we replace φ_k by a conjunction of working formulas. Thus, let $\bigwedge_{j \in J_k} \phi_j$ be a conjunction of working formulas which is equivalent to φ_k in T. The conjunction of working formulas $\bigwedge_{i \in I} \varphi_i$ is equivalent in T to

$$(\bigwedge_{i \in I - \{k\}} \varphi_i) \wedge (\bigwedge_{j \in J_k} \phi_j)$$

which is clearly a conjunction of working formulas.

2. Or, we replace a strict sub-working formula of φ_k by a conjunction of working formulas. Thus, let ϕ be a sub-working formula of φ_k of depth $d_\phi < d_k$ (thus ϕ is different from φ_k). Thus, φ_k has a sub-working formula[10] of the form

$$\neg(\exists \bar{x}\alpha \wedge (\bigwedge_{l \in L} \psi_l) \wedge (\phi)),$$

where L is a finite (possibly empty) set and all the ψ_l are working formulas. Let $\bigwedge_{j \in J} \phi_j$ be a conjunction of working formulas which is equivalent to ϕ in T. Thus the preceding sub-working formula of φ_k is equivalent in T to

$$\neg(\exists \bar{x}\alpha \wedge (\bigwedge_{l \in L} \psi_l) \wedge (\bigwedge_{j \in J} \phi_j)),$$

which is clearly a sub-working formula and thus φ_k is equivalent to a working formula and thus $\bigwedge_{i \in I} \varphi_i$ is equivalent to a conjunction of working formulas.

[10] By considering that the set of the sub-formulas of any formula φ contains also the whole formula φ.

From 1 and 2 we deduce that (i) every substitution of a sub-working formula of a conjunction of working formulas by a conjunction of working formulas produces a conjunction of working formulas.

Since each rule transforms a working formula into a conjunction of working formulas, then according to (i) every finite application of the rewriting rules on a working formula produces a conjunction of working formulas. Let us show now that each of these final working formulas is solved.

Let φ be a working formula. Let ϕ be the conjunction of working formulas obtained after finite application of the rules on φ. Suppose that the rules can not be applied anymore and one of the working formula of ϕ is not solved. Let ψ be this formula, two cases arise:

Case 1: ψ is a working formula of depth greater than 2. Thus, ψ has a sub-formula of the form

$$\neg\left[\begin{array}{l}\exists\bar{x}\,\alpha\wedge\psi_1\wedge\\ \neg\left[\ \exists\bar{y}\,\beta\wedge\bigwedge_{i\in I}\neg(\exists\bar{z}_i\,\delta_i)\ \right]\end{array}\right]$$

where ψ_1 is a conjunction of working formulas, I is a nonempty set and α, β and δ_i are elements of A for all $i\in I$. Let $(\exists\bar{y}'\beta'\wedge(\exists\bar{x}''\beta''\wedge(\exists\bar{y}'''\beta''')))$ be the decomposed formula in T of $\exists\bar{y}\beta$ and let $(\exists\bar{z}_i'\delta_i'\wedge(\exists\bar{z}_i''\delta_i''\wedge(\exists\bar{z}_i'''\delta_i''')))$ be the decomposed formula in T of $\exists\bar{z}_i\delta_i$. If $\exists\bar{y}'''\beta'''$ is not the formula $\exists\varepsilon\,true$ then the rule (3) can still be applied which contradicts our supposition. Thus, suppose that

$$\exists\bar{y}'''\beta'''=\exists\varepsilon\,true \tag{2.27}$$

If there exists $k\in I$ such that $\exists\bar{z}_k'''\delta_k'''$ is not the formula $\exists\varepsilon\,true$ then the rule (3) can be still applied (with $I=\emptyset$) which contradicts our supposition. Thus, suppose that

$$\exists\bar{z}_i'''\delta_i'''=\exists\varepsilon\,true \tag{2.28}$$

for all $i\in I$. If there exists $k\in I$ such that $\exists\bar{z}_k\delta_k$ is not an element of A' then since we have (2.28), the rule (4) can still be applied (with $I=\emptyset$) which contradicts our supposition. Thus, suppose that

$$\exists\bar{z}_i\delta_i\in A' \tag{2.29}$$

for all $i\in I$. If $\exists\bar{y}\beta$ is not an element of A' then since we have (2.27) and (2.29), the rule (4) can still be applied which contradicts our supposition. Thus, suppose that

$$\exists\bar{y}\beta\in A' \tag{2.30}$$

Since we have (2.29) and (2.30) then the rule (5) can still be applied which contradicts all our suppositions.

Case 2: ψ is a working formula of the form

$$\neg(\exists\bar{x}\,\alpha\wedge\bigwedge_{i\in I}\neg(\exists\bar{y}_i\,\beta_i))$$

where at least one of the following conditions holds:

1. α is the formula *false*,
2. there exists $k\in I$ such that β_k is the formula *true* or *false*,
3. there exists $k\in I$ such that $\exists\bar{y}_k\beta_k\notin A'$,

4. $\exists \bar{x}\alpha \notin A'$.

If the condition (1) holds then the rule (2) can still be applied which contradicts our suppositions. If the condition (2) holds then the rules (1) and (2) can still be applied which contradicts our suppositions. If the condition (3) holds then the rule (3) or (4) (with $I = \emptyset$) can still be applied which contradicts our suppositions. If the condition (4) holds then according to the preceding point $\exists \bar{y}_i \beta_i \in A'$ for all $i \in I$ and thus the rule (3) or (4) can still be applied which contradicts our suppositions.

From Case 1 and Case 2, our suppositions are always false thus ψ is a solved formula and thus ϕ is a conjunction of solved formulas.

2.3.4 The decision procedure

Having any formula ψ, the resolution of ψ proceeds as follows:

1. Transform the formula ψ into a normalized formula and then into a working formula φ which is wnfv and equivalent to ψ in T.

2. Apply the preceding rewriting rules on φ as many time as possible. At the end we obtain a conjunction ϕ of solved formulas.

According to Property 2.3.3.1, the application of the rewriting rules on a formula ψ without free variables produces a conjunction ϕ of solved formulas which is equivalent to ψ in T and does not contain free variables. According to Property 2.3.2.5, ϕ is either the formula *true* or $\neg$*true*, thus either $T \models \psi$ or $T \models \neg\psi$ and thus T is a complete theory.

Corollary 2.3.4.1 *If T is infinite-decomposable then every formula is equivalent in T either to true or to false or to a Boolean combination of elements of A' which has at least one free variable.*

2.4 Application to the theory $\mathcal{T}$ of finite or infinite trees

2.4.1 The axioms of $\mathcal{T}$

The theory $\mathcal{T}$ of finite or infinite trees built on an **infinite** set F of distinct function symbols has as axioms the infinite set of propositions of one of the three following forms:

$$\begin{array}{lll} \forall\bar{x}\forall\bar{y} & \neg f\bar{x} = g\bar{y} & [1] \\ \forall\bar{x}\forall\bar{y} & f\bar{x} = f\bar{y} \rightarrow \bigwedge_i x_i = y_i & [2] \\ \forall\bar{x}\exists!\bar{z} & \bigwedge_i z_i = t_i[\bar{x}\bar{z}] & [3] \end{array}$$

where f and g are distinct function symbols taken from F, $\bar{x}$ is a vector of possibly non-distinct variables x_i, $\bar{y}$ is a vector of possibly non-distinct variables y_i, $\bar{z}$ is a vector of distinct variables z_i and $t_i[\bar{x}\bar{z}]$ is a term which begins with an element of F followed by variables taken from $\bar{x}$ or $\bar{z}$. Note that this theory does not admit full elimination of quantifiers. In fact, in the formula $\exists x\, y = f(x)$ we can not remove or eliminate the quantifier $\exists x$.

2.4.2 Properties of $\mathcal{T}$

Suppose that the variables of V are ordered by a strict linear dense order relation without endpoints denoted by $\succ$.

Definition 2.4.2.1 *A conjunction α of flat equations is called* $(\succ)$-solved *if all its left-hand sides are distinct and α does not contain equations of the form $x = x$ or $y = x$, where x and y are variables such that $x \succ y$.*

Property 2.4.2.2 *Every conjunction α of flat formulas is equivalent in $\mathcal{T}$ either to false or to a $(\succ)$-solved conjunction of flat equations.*

Proof. To prove this property we introduce the following rewriting rules:

(1)	$false \wedge \alpha$	$\Longrightarrow$	$false,$
(2)	$x = fy_1...y_m \wedge x = gz_1...z_n$	$\Longrightarrow$	$false,$
(3)	$x = fy_1...y_n \wedge x = fz_1...z_n$	$\Longrightarrow$	$x = fy_1...y_n \wedge \bigwedge_{i \in \{1,...,n\}} y_i = z_i,$
(4)	$x = x$	$\Longrightarrow$	$true$
(5)	$y = x$	$\Longrightarrow$	$x = y$
(6)	$x = y \wedge x = fz_1...z_n$	$\Longrightarrow$	$x = y \wedge y = fz_1...z_n$
(7)	$x = y \wedge x = z$	$\Longrightarrow$	$x = y \wedge y = z$

with α any formula and f and g two distinct function symbols taken from F. The rules (5), (6) and (7) are applied only if $x \succ y$. This condition prevents infinite loops.

Let us prove now that every repeated application of the preceding rewriting rules on any conjunction α of flat formulas, terminates and producing either the formula *false* or a $(\succ)$-solved conjunction of flat equations which is equivalent to α in $\mathcal{T}$.

Proof, first part: The application of the rewriting rules terminates. Since the variables which occur in our formulas are ordered by the strict linear order relation without endpoints " $\succ$ ", we can number them by positive integers such that

$$x \succ y \leftrightarrow no(x) > no(y),$$

where $no(x)$ is the number associated to the variable x. Let us consider the 4-tuples (n_1, n_2, n_3, n_4) where the n_i's are the following positive integers:

- n_1 is the number of occurrences of sub-formulas of the form $x = fy_1...y_n$, with $f \in F$,
- n_2 is the number of occurrences of atomic formulas,
- n_3 is the sum of the $no(x)$'s for all occurrence of a variable x,
- n_4 is the number of occurrences of formulas of the form $y = x$, with $x \succ y$.

For each rule, there exists a row i such that the application of this rule decreases or does not change the value of the n_j's, with $1 \leq j < i$, and decreases the value of n_i. The row i is equal to: 2 for the rule (1), 1 for the rules (2) and (3), 3 for the rules (4), (6) and (7), 4 for the rule (5). To each sequence of formulas obtained by a finite application of the preceding rewriting rules, we can associate a series of 4-tuples (n_1, n_2, n_3, n_4) which is strictly decreasing in the lexicographic order. Since the n_i's are positive integers, they cannot be negative, thus, this series of 4-tuples is a finite series and the application of the rewriting rules terminates.

Proof, second part: The rules preserve equivalence in $\mathcal{T}$. The rule (1) is evident in $\mathcal{T}$. The rules (2) preserves the equivalence in $\mathcal{T}$ according to the axiom 1. The rule (3) preserves the equivalence in $\mathcal{T}$ according to the axiom 2. The rules (4), (5), (6) and (7) are evident in $\mathcal{T}$.

Proof, third part: The application of the rewriting rules terminates either by *false* or by a ($\succ$)-solved conjunction of flat equations. Suppose that the application of the rewriting rules on a conjunction α of flat formulas terminates by a formula β and at least one of the following conditions holds:

1. β is not the formula *false* and has at least a sub-formula of the form *false*,
2. β has two equations with the same left-hand side,
3. β contains equations of the form $x = x$ or $y = x$ with $x \succ y$.

If the condition 1 holds then the rule (1) can still be applied which contradicts our supposition. If the condition 2 holds then the rules (2), (3), (6) and (7) can still be applied which contradicts our supposition. If the condition 3 holds then the rules (4) and (5) can still be applied which contradicts our supposition. Thus, the formula β according to Definition 2.4.2.1 is either the formula *false* or a ($\succ$)-solved conjunction of flat equations. □

Let us introduce now the notion of *reachable variable* and *reachable equation.*

Definition 2.4.2.3 *The equations and reachable variables from the variable u in the formula*

$$\exists \bar{x} \bigwedge_{i=1}^{n} v_i = t_i$$

are those which occur in at least one of its sub-formulas of the form $\bigwedge_{j=1}^{m} v_{k_j} = t_{k_j}$, where v_{k_1} is the variable u and v_{k_j+1} occurs in the term t_{k_j} for all $j \in \{1, .., m\}$. The equations and reachable variables of this formula are those who are reachable from a variables which does not occur in $\bar{x}$.

Example 2.4.2.4 *In the formula*

$$\exists uvw\, z = fuv \wedge v = gvu \wedge w = fuv,$$

the equations $z = fuv$ and $v = gvu$ and the variables u and v are reachable. On the other hand the equation $w = fuv$ and the variable w are not reachable.

According to the axioms [1] and [2] of $\mathcal{T}$ we have the following property

Property 2.4.2.5 *Let α be a conjunction of flat equations. If all the variables of $\bar{x}$ are reachable in $\exists \bar{x}\, \alpha$ then $\mathcal{T} \models \exists ? \bar{x}\, \alpha$.*

According to the axiom 3 we have:

Property 2.4.2.6 *Let α be a ($\succ$)-solved conjunction of flat equations and let $\bar{x}$ be the vector of its left-hand sides. We have $\mathcal{T} \models \exists ! \bar{x}\, \alpha$.*

2.4.3 $\mathcal{T}$ is infinite-decomposable

Property 2.4.3.1 *$\mathcal{T}$ is a decomposable theory.*

Let us show that $\mathcal{T}$ satisfies the conditions of Definition 2.2.1.1.

Choice of the sets $\Psi(u)$, A, A', A'' and A'''

Let F_0 be the set of the 0-ary function symbols of F. The sets $\Psi(u)$, A, A', A'' and A''' are chosen as follows:

- $\Psi(u)$ is the set $\{faux\}$ if $F - F_0 = \emptyset$, else it contains formulas of the form $\exists \bar{y}\, u = f\bar{y}$ with $f \in F - F_0$,
- A is the set FL,
- A' is the set of the formulas of the form $\exists \bar{x}' \alpha'$ such that
  - α' is either the formula *false* or a $(\succ)$-solved conjunction of flat equations where the order $\succ$ is such that all the variables of $\bar{x}'$ are greater than the free variables of $\exists \bar{x}' \alpha'$,
  - all the variables of $\bar{x}'$ and all the equations of α' are reachable in $\exists \bar{x}' \alpha'$,
- A'' is the set of the formulas of the form $\exists \bar{x}''\, \mathit{true}$,
- A''' is the set of the formulas of the form $\exists \bar{x}''' \alpha'''$ such that α''' is a $(\succ)$-solved conjunction of flat equations and $\bar{x}'''$ is the vector of the left-hand sides of the equations of α'''.

It is clear that FL is $\mathcal{T}$-closed and A', A'' and A''' contain formulas of the form $\exists \bar{x}\, \alpha$ with $\alpha \in FL$. Let us show now that $\mathcal{T}$ satisfies the five condition of Definition 2.2.1.1

$\mathcal{T}$ satisfies the first condition

Let us show that every formula of the form $\exists \bar{x}\, \alpha \wedge \psi$, with $\alpha \in FL$ and ψ any formula, is equivalent in $\mathcal{T}$ to a wnfv formula of the form

$$\exists \bar{x}'\, \alpha' \wedge (\exists \bar{x}''\, \alpha'' \wedge (\exists \bar{x}'''\, \alpha''' \wedge \psi)), \tag{2.31}$$

with $\exists \bar{x}'\, \alpha' \in A'$, $\exists \bar{x}''\, \alpha'' \in A''$ and $\exists \bar{x}'''\, \alpha''' \in A'''$.

Let us choose the order $\succ$ such that all the variables of $\bar{x}$ are greater than the free variables of $\exists \bar{x}\alpha$. According to Property 2.4.2.2 two cases arise:

Either α is equivalent to *false* in $\mathcal{T}$. Thus, $\bar{x}' = \bar{x}'' = \bar{x}''' = \varepsilon$, $\alpha' = \mathit{false}$ and $\alpha'' = \alpha''' = \mathit{true}$.

Or, α is equivalent to a $(\succ)$-solved conjunction β of flat equations. Let X be the set of the variables of the vector $\bar{x}$. Let Y_{rea} be the set of the reachable variables of $\exists \bar{x}\beta$. Let Lhs be the set of the variables which occur in a left-hand side of an equation of β. We have:
− $\bar{x}'$ contains the variables of $X \cap Y_{rea}$.
− $\bar{x}''$ contains the variables of $(X - Y_{rea}) - Lhs$.
− $\bar{x}'''$ contains the variables of $(X - Y_{rea}) \cap Lhs$.
− α' is the conjunction of the reachable equations of $\exists \bar{x}\beta$.
− α'' is the formula *true*.
− α''' is the conjunction of the unreachable equations of $\exists \bar{x}\beta$.

According to our construction it is clear that $\exists \bar{x}'\alpha' \in A'$, $\exists \bar{x}''\alpha'' \in A''$ and $\exists \bar{x}'''\alpha \in A'''$. Let us show that (2.31) and $\exists \bar{x}\alpha \wedge \psi$ are equivalent in $\mathcal{T}$. Let X', X'' and X''' be the sets of the variables of the vectors $\bar{x}'$, $\bar{x}''$ and $\bar{x}'''$. If α is equivalent to *false* in $\mathcal{T}$ then the equivalence of the decomposition is evident. Else β is a conjunction of flat equations and thus according to our construction we have: $X = X' \cup X'' \cup X'''$, $X' \cap X'' = \emptyset$, $X' \cap X''' = \emptyset$, $X'' \cap X''' = \emptyset$, for all $x''_i \in X''$ we have $x''_i \notin var(\alpha')$ and for all $x'''_i \in X'''$ we have $x'''_i \notin var(\alpha' \wedge \alpha'')$. Moreover each equation of β occurs in $\alpha' \wedge \alpha'' \wedge \alpha'''$ and each equation in $\alpha' \wedge \alpha'' \wedge \alpha'''$ occurs in β and thus $\mathcal{T} \models \beta \leftrightarrow (\alpha' \wedge \alpha'' \wedge \alpha''')$. We have shown that the vectorial quantifications are coherent and the equivalence $\mathcal{T} \models \beta \leftrightarrow \alpha' \wedge \alpha'' \wedge \alpha'''$ holds. According to Property 2.4.2.2, we have $\mathcal{T} \models \alpha \leftrightarrow \beta$ and thus, the decomposition keeps the equivalence in $\mathcal{T}$.

Example 2.4.3.2 *Let us decompose the following formula φ*

$$\exists xyv\, z = fxy \wedge z = fxw \wedge v = fz.$$

First, since w and z are free in φ then the order $\succ$ will be chosen as follows:

$$x \succ y \succ v \succ w \succ z.$$

Note that the quantified variables are greater than the free variables. Then, using the rewriting rules of Property 2.4.2.2 we transform the conjunction of equations into a ($\succ$)-solved formula. Thus, the formula φ is equivalent in $\mathcal{T}$ to the following formula ψ

$$\exists xyv\, z = fxy \wedge y = w \wedge v = fz.$$

Since the variables x, y, w and the equations $z = fxy, y = w$ are reachable in ψ, then ψ is equivalent in $\mathcal{T}$ to the following decomposed formula

$$\exists xy\, z = fxy \wedge y = w \wedge (\exists \varepsilon\, true \wedge (\exists v\, v = fz)).$$

It is clear that $(\exists xy\, z = fxy \wedge y = w) \in A'$, $(\exists \varepsilon\, true) \in A''$ and $(\exists v\, v = fz) \in A'''$.

$\mathcal{T}$ satisfies the second condition

Let us show that if $\exists \bar{x}'\alpha' \in A'$ then $\mathcal{T} \models \exists ?\bar{x}'\alpha'$. Since $\exists \bar{x}'\alpha' \in A'$ and according to the choice of the set A', either α' is the formula *false* and thus we have immediately $\mathcal{T} \models \exists ?\bar{x}'\alpha'$ or α' is a ($\succ$)-solved conjunction of flat equations and the variables of $\bar{x}'$ are reachable in $\exists \bar{x}'\alpha'$. Thus, using Property 2.4.2.5 we get $\mathcal{T} \models \exists ?\bar{x}'\alpha'$.

Let us show now that if y is a free variable of $\exists \bar{x}'\alpha'$ then $\mathcal{T} \models \exists ?y\bar{x}'\,\alpha'$ or there exists $\psi(u) \in \Psi(u)$ such that $\mathcal{T} \models \forall y\, (\exists \bar{x}'\,\alpha') \rightarrow \psi(y)$. Let y be a free variable of $\exists \bar{x}'\alpha'$. It is clear that α' can not be in this case the formula *false*. Thus, four cases arise:

If y occurs in a sub-formula of α' of the form $y = t(\bar{x}', \bar{z}', y)$, where $\bar{z}'$ is the set of the free variables of $\exists \bar{x}'\alpha'$ which are different from y and where $t(\bar{x}', \bar{z}', y)$ is a term which begins by an element of $F - F_0$ followed by variables taken from $\bar{x}'$ or $\bar{z}'$ or $\{y\}$, then the formula $\exists \bar{x}'\alpha'$ implies in $\mathcal{T}$ the formula $\exists \bar{x}'\, y = t(\bar{x}', \bar{z}', y)$, which implies in $\mathcal{T}$ the formula $\exists \bar{x}'\bar{z}'w\, y = t(\bar{x}', \bar{z}', w)$, where $y = t(\bar{x}', \bar{z}', w)$ is the formula $y = t(\bar{x}', \bar{z}', y)$ in which we have replaced every free occurrence of y in the term $t(\bar{x}', \bar{z}', y)$ by the variable w. According to the choice of the set $\Psi(u)$, the formula $\exists \bar{x}'\bar{z}'w\, u = t(\bar{x}', \bar{z}', w)$ belongs to $\Psi(u)$.

If y occurs in a sub-formula of α' of the form $y = f_0$ with $f_0 \in F_0$ then $\mathcal{T} \models \exists !y\, y = f_0$ according to the third axiom of $\mathcal{T}$. Thus (i) $\mathcal{T} \models \exists ?y\,\alpha'$. On the other hand, since α' is ($\succ$)-solved, y has no occurrences in an other left-hand side of an equation of α', thus, since the variables of $\bar{x}$ are reachable in $\exists \bar{x}'\alpha'$ (according to the choice of the set A'), all the variables of $\bar{x}'$ keep reachable in $\exists \bar{x}'y\,\alpha'$ and thus using (i) and Property 2.4.2.5 we get $\mathcal{T} \models \exists ?\bar{x}'y\,\alpha'$.

If y occurs in a sub-formula of α' of the form $y = z$ then:

1. According to the choice of the set A', the order $\succ$ is such that all the variables of $\bar{x}'$ are greater than the free variables of $\exists \bar{x}'\alpha'$.

2. According to Definition 2.4.2.2 of the ($\succ$)-solved formula, we have $y \succ z$.

From (1) and (2), we deduce that z is a free variable in $\exists \bar{x}' \alpha'$. Since α' is ($\succ$)-solved, y has no occurrences in an other left-hand side of an equation of α', thus, since the variables of $\bar{x}$ are reachable in $\exists \bar{x}' \alpha'$ (according to the choice of the set A'), all the variables of $\bar{x}'$ keep reachable in $\exists \bar{x}' y\, \alpha'$. Moreover, for each value of z there exists at most a value for y. Thus, using Property 2.4.2.5 we get $\mathcal{T} \models \exists ? \bar{x}' y\, \alpha'$.

If y occurs only in the right-hand sides of the equations of α', then according to the choice of the set A', all the variables of $\bar{x}'$ and all the equations of α' are reachable in $\exists \bar{x}' \alpha'$. Thus, since y does not occur in a left-hand side of an equation of α', the variable y and the variables of $\bar{x}'$ are reachable in $\exists \bar{x}' y\, \alpha'$ and thus using Property 2.4.2.5 we get $\mathcal{T} \models \exists ? \bar{x}' y\, \alpha'$. In all the cases $\mathcal{T}$ satisfies the second condition of Definition 2.2.1.1.

$\mathcal{T}$ satisfies the third condition

First, we present a property which holds in any model M of $\mathcal{T}$. This property results from the axiomatization of $\mathcal{T}$ (more exactly from the axioms 1 and 2) and the infinite set of function symbols F.

Property 2.4.3.3 *Let M be a model of $\mathcal{T}$ and let f be a function symbol taken from $F - F_0$. The set of the individuals i of M, such that $M \models \exists \overline{x}\, i = f\overline{x}$, is infinite.*

Let $\exists \bar{x}'' \alpha''$ be a formula which belongs to A''. According to the choice of A'', this formula is of the form $\exists \bar{x}''$ *true*. Let us show that, for every variable x''_j of $\bar{x}''$ we have $\mathcal{T} \models \exists_{\infty}^{\Psi(u)} x_j$ *true*. Two cases arise:

Either $F - F_0 = \emptyset$ then $\Psi(u) = \{faux\}$ and F_0 is infinite since the theory is defined on an infinite set of function symbols. According to Axiom of conflict of symbols, for every distinct constants f and g correspond two distinct individuals in every model of $\mathcal{T}$. Thus, since F_0 is infinite there exists an infinite set of individuals in every model of $\mathcal{T}$ and thus according to Definition 2.1.2.1, we have: $\mathcal{T} \models \exists_{\infty}^{\{false\}} x_j$ *true*.

Or, $F - F_0 \neq \emptyset$, thus $\Psi(u)$ contains formulas of the form $\exists \bar{z}\, u = f\bar{z}$ with $f \in F - F_0$. Let M be a model of $\mathcal{T}$. Since the formula $\exists x''_j$ *true* does not have free variables, it is already instantiated, and thus according to Definition 2.1.2.1 it is enough to show that there exists an infinity of individuals i of M which satisfy the following condition:

$$M \models \neg\psi_1(i) \wedge \cdots \wedge \neg\psi_n(i), \tag{2.32}$$

with $\psi_j(u) \in \Psi(u)$, i.e. of the form $\exists \bar{z}\, u = f\bar{z}$ with $f \in F - F_0$. Two cases arise:

- Either $F - F_0$ is a finite set, then F_0 is infinite because the theory is defined on an infinite set of function symbols. Thus, there exists an infinity of constants f_k which are different from all the function symbols of all the $\psi_j(u)$ and thus using the axiom of conflict of symbols there exists an infinity of distinct individuals i such that (2.32).

- Or, $F - F_0$ is infinite, then there exists a formula $\psi(u)^* \in \Psi(u)$ which is different from all the $\psi_j(u)$ of (2.32), i.e. which has a function symbol which is different from the function symbols of all the $\psi_1(u) \cdots \psi_n(u)$. According to Property 2.4.3.3 there exists an infinity of individuals i such that $M \models \psi(i)^*$. Since this $\psi(u)^*$ is different from all the $\psi_j(u)$, then according to axiom of conflict of symbols there exists an infinite set of individuals i such that $M \models \psi(i)^* \wedge \neg\psi_1(i) \wedge \cdots \wedge \neg\psi_n(i)$ and thus such that (2.32).

$\mathcal{T}$ satisfies the fourth condition

Let us show that if $\exists \bar{x}'''\alpha''' \in A'''$ then $\mathcal{T} \models \exists! \bar{x}'''\, \alpha'''$. Let $\exists \bar{x}'''\alpha'''$ be an element of A'''. According to the choice of the set A''' and Property 2.4.2.6 we get immediately $\mathcal{T} \models \exists! \bar{x}'''\alpha'''$.

$\mathcal{T}$ satisfies the fifth condition

Let us show that if the formula $\exists \bar{x}'\alpha'$ belongs to A' and has no free variables then this formula is either the formula $\exists \varepsilon\, \mathit{true}$ or $\exists \varepsilon\, \mathit{false}$. Let $\exists \bar{x}'\alpha'$ be a formula, without free variables, which belongs to A'. We have

1. According to the choice of the set A', all the variables and equations of $\exists \bar{x}'\alpha'$ are reachable in $\exists \bar{x}'\alpha'$ and α' is either the formula *false* or a $(\succ)$-solved conjunction of flat equations.

2. Since the formula $\exists \bar{x}'\alpha'$ has no free variables and according to Definition 2.4.2.3 there exists in this case neither variables nor equations reachable in $\exists \bar{x}'\alpha'$,

Thus, from (1) and (2), $\bar{x}'$ is the empty vector, i.e. ε and α' is either the formula *true* or *false*.

We have shown that $\mathcal{T}$ satisfies the five conditions of Definition 2.2.1.1. Moreover, T. Dao has shown in [16] that this theory has as model the algebra of finite or infinite trees introduced by Maher in [33], then $\mathcal{T}$ is infinite-decomposable and thus complete. □

2.4.4 Solving first-order propositions in $\mathcal{T}$

Example 2.4.4.1 *Let us solve the following formula φ_1 in $\mathcal{T}$:*

$$\exists x \forall y\, ((\exists zwv\, y = fz \wedge y = fx \wedge w = gzv) \vee (x = fy \wedge x = fx))$$

Using Property 2.3.1.3 we first transform the preceding formula into the following normalized formula

$$\neg(\exists \varepsilon\, \mathit{true} \wedge \neg(\exists x\, \mathit{true} \wedge \neg \begin{bmatrix} \exists y\, \mathit{true} \wedge \\ \neg(\exists zwv\, y = fz \wedge y = fx \wedge w = gzv) \wedge \\ \neg(\exists \varepsilon\, x = fy \wedge x = fx) \end{bmatrix})) \tag{2.33}$$

Since $A = FL$ then the preceding normalized formula is a working formula. Let us decompose the sub-formula

$$\exists zwv\, y = fz \wedge y = fx \wedge w = gzv. \tag{2.34}$$

According to Section 2.4.3, the order $\succ$ is chosen such that $z \succ w \succ v \succ y \succ x$. Using the rewriting rules of Property 2.4.2.2, the sub-formula $y = fz \wedge y = fx \wedge w = gzv$ is equivalent in $\mathcal{T}$ to the $(\succ)$-solved formula $y = fz \wedge z = x \wedge w = gzv$, and thus according to Section 2.4.3, the decomposed formula of (2.34) is

$$\exists z\, y = fz \wedge z = x \wedge (\exists v\, \mathit{true} \wedge (\exists w\, w = gzv))$$

Since $(\exists w\, w = gzv) \neq (\exists \varepsilon\, \mathit{true})$ we can apply the rule (3) with $I = \emptyset$, thus, the formula (2.33) is equivalent in $\mathcal{T}$ to

$$\neg(\exists \varepsilon\, \mathit{true} \wedge \neg(\exists x\, \mathit{true} \wedge \neg \begin{bmatrix} \exists y\, \mathit{true} \wedge \\ \neg(\exists zv\, y = fz \wedge z = x) \wedge \\ \neg(\exists \varepsilon\, x = fy \wedge x = fx) \end{bmatrix})) \tag{2.35}$$

The sub-formula $\exists zv\, y = fz \wedge z = x$ *is not an element of* A' *and is equivalent in* $\mathcal{T}$ *to the decomposed formula* $\exists z\, y = fz \wedge z = x \wedge (\exists v\, true \wedge (\exists \varepsilon\, true))$*, thus we can apply the rule (4) with* $I = \emptyset$ *and the formula (2.35) is equivalent in* $\mathcal{T}$ *to*

$$\neg(\exists \varepsilon\, true \wedge \neg(\exists x\, true \wedge \neg \begin{bmatrix} \exists y\, true \wedge \\ \neg(\exists z\, y = fz \wedge z = x) \wedge \\ \neg(\exists \varepsilon\, x = fy \wedge x = fx) \end{bmatrix})) \tag{2.36}$$

Let us decompose now the sub-formula

$$\exists \varepsilon\, x = fy \wedge x = fx \tag{2.37}$$

Using the rewriting rules of Property 2.4.2.2, the sub-formula $x = fy \wedge x = fx$ *is equivalent in* $\mathcal{T}$ *to the* $(\succ)$*-solved formula* $x = fy \wedge y = x$ *and thus according to Section 2.4.3 the decomposed formula of (2.37) is*

$$\exists \varepsilon\, x = fy \wedge y = x \wedge (\exists \varepsilon\, true \wedge (\exists \varepsilon\, true))$$

Since $(\exists \varepsilon\, x = fy \wedge x = fx) \notin A'$*, then we can apply the rule (4) with* $I = \emptyset$ *and thus the formula (2.36) is equivalent in* $\mathcal{T}$ *to*

$$\neg(\exists \varepsilon\, true \wedge \neg(\exists x\, true \wedge \neg \begin{bmatrix} \exists y true \wedge \\ \neg(\exists z\, y = fz \wedge z = x) \wedge \\ \neg(\exists \varepsilon\, x = fy \wedge y = x) \end{bmatrix})) \tag{2.38}$$

According to Section 2.4.3, the formula $\exists \varepsilon\, vrai \wedge (\exists y\, true \wedge (\exists \varepsilon\, true))$ *is the decomposed formula of* $\exists y\, true$*. Since* $\exists y\, true \notin A'$*,* $(\exists z\, y = fz \wedge z = x) \in A'$ *and* $(\exists \varepsilon\, x = fy \wedge y = x) \in A'$ *then we can apply the rule (4) and thus the formula (2.38) is equivalent in* $\mathcal{T}$ *to*

$$\neg(\exists \varepsilon\, true \wedge \neg(\exists \varepsilon\, true \wedge \neg(\exists \varepsilon\, true)) \tag{2.39}$$

Finally, we can apply the rule (1) thus the formula (2.39) is equivalent in $\mathcal{T}$ *to* $\neg(\exists \varepsilon\, true)$*. Thus* φ_1 *is false in* $\mathcal{T}$*.*

Example 2.4.4.2 *Let us solve the following formula* φ_2 *in* $\mathcal{T}$*:*

$$\exists x\, \forall y\, ((\exists z\, y = fz \wedge z = x) \vee (\exists \varepsilon\, x = fy \wedge y = x) \vee \neg(x = fy)) \tag{2.40}$$

Using Property 2.3.1.3 we first transform the preceding formula into the following normalized formula

$$\neg(\exists \varepsilon\, true \wedge \neg(\exists x\, true \wedge \neg \begin{bmatrix} \exists y\, x = fy \wedge \\ \neg(\exists z\, y = fz \wedge z = x) \wedge \\ \neg(\exists \varepsilon\, x = fy \wedge y = x) \end{bmatrix})) \tag{2.41}$$

Since $A = FL$ *then the preceding normalized formula is a working formula in* $\mathcal{T}$*. Since* $(\exists y\, x = fy) \in A'$*,* $(\exists z\, y = fz \wedge z = x) \in A'$ *and* $(\exists \varepsilon\, x = fy \wedge y = x) \in A'$ *then we can apply the rule (5), thus the formula (2.41) is equivalent in* $\mathcal{T}$ *to*

$$\neg \begin{bmatrix} \exists \varepsilon\, true \wedge \\ \neg(\exists x\, true \wedge \neg(\exists y\, x = fy)) \wedge \\ \neg(\exists x_1 y_1 z\, x_1 = fy_1 \wedge y_1 = fz \wedge z = x_1) \wedge \\ \neg(\exists x_2 y_2\, x_2 = fy_2 \wedge x_2 = fy_2 \wedge y_2 = x_2) \end{bmatrix} \tag{2.42}$$

According to Section 2.4.3, the formula $\exists\varepsilon\, true \wedge (\exists x\, true \wedge (\exists\varepsilon\, true))$ *is the decomposed formula of* $\exists x\, true$*. Since* $(\exists x\, true) \notin A'$ *and* $(\exists y\, x = fy) \in A'$ *then we can apply the rule (4) and thus the formula (2.42) is equivalent in* $\mathcal{T}$ *to*

$$\neg\begin{bmatrix} \exists\varepsilon\, true\wedge \\ \neg(\exists\varepsilon\, true)\wedge \\ \neg(\exists x_1 y_1 z\, x_1 = fy_1 \wedge y_1 = fz \wedge z = x_1)\wedge \\ \neg(\exists x_2 y_2\, x_2 = fy_2 \wedge x_2 = fy_2 \wedge y_2 = x_2) \end{bmatrix} \tag{2.43}$$

Finally, we can apply the rule (1), thus the formula (2.43) is equivalent in $\mathcal{T}$ *to true. Thus* φ_2 *is true in* $\mathcal{T}$*.*

2.5 Discussion and partial conclusion

Our decision procedure which is ideal for deciding the validity of complex propositions can also be applied to first order formulas having free variables and produces in this case a conjunction ϕ of solved formulas easily transformable into a boolean combination of basic formulas. But in no cases, our algorithm can warrant that ϕ is neither true nor false if it contains at least one free variables. It can not also present the solutions of the free variables in a clear and explicit way and can not detect if a formula having at least one free variable is always true or false. This is why this algorithm is called *decision procedure* and not *general algorithm solving first order constraints.*

In the other hand, we have shown the infinite-decomposability of fundamental theories such as: the equational theory, the theory of additive rational or real numbers, the theory of finite trees, the theory of infinite trees, the theory of finite or infinite trees and a combination of finite or infinite trees with additive rational or real numbers [26]. What about the decomposability of the theory of linear dense order ? If we take as model the rational numbers then for every instantiation of the free variables of the formula $\exists x\, z < x \wedge x < y$: either there exists an infinity values for x, or there exists no values for x ! In fact, if the variables z and y are instantiated respectively by 1 and 0, then there exists no instantiations for x such that $1 < x \wedge x < 0$! This new behavior does not satisfy the infinite quantifier and thus the theory of linear dense order is not infinite-decomposable. We must find a new quantifier more expressive than the infinite quantifier ! This will be our goal in Chapter 3.

Chapter 3

Zero-infinite-decomposable theory

Contents

We present in this chapter the class of the *zero-infnite-decomposable theories* which is an extension of the infinite-decomposable theories, and where the infinite quantifier has been replaced by a new quantifier called *zero-infinite*. We show the completeness of these theories using the sufficient condition of completeness of first order theories given in Chapter 1, and give some examples of fundamental zero-infinite-decomposable theories. We present also a property that links the infinite-decomposable theories to the zero-infinite-decomposable theories and show that the infinite theories Eq, Ra and $\mathcal{T}$ are also zero-infinite-decomposable. Then, we give a decision procedure in every zero-infinite-decomposable theory T, in the form of six rewriting rules which transform a formula φ, which can possibly contain free variables, into a wnfv conjunction ϕ of solved formulas, equivalent to φ in T and such that ϕ is, either the formula *true*, or the formula $\bigwedge_{i \in I} \neg true$, or a formula having at least one free variable and being easily transformable into a boolean combination of quantified conjunctions of atomic formulas. In particular, if φ has no free variables, then ϕ is, either the formula *true*, or the formula $\neg true$. The correctness of our algorithm is another proof of the completeness of the zero-infinite-decomposable theories. We end this chapter by an application to the construction of trees on an ordered set. This theory denoted by $\mathcal{T}_{ord}$, is a complete axiomatization of a tree construction on any set of individuals together with a linear dense order relation without endpoints. After having presented the axioms of $\mathcal{T}_{ord}$, we show its zero-infinite-decomposability and end by an example on solving propositions in $\mathcal{T}_{ord}$. Note that the results presented in this chapter have been published in [23], [24] and [25].

3.1 Zero-infinite quantifier: $\exists_{o\,\infty}^{\Psi(u)}$

Let M be a model and T a theory. Let $\Psi(u)$ be a set of formulas having at most one free variable u. Let φ and φ_j be M-formulas.

Definition 3.1.0.3 *We write*

$$M \models \exists_{o\,\infty}^{\Psi(u)} x\, \varphi(x), \tag{3.1}$$

if for each instantiation $\exists x\, \varphi'(x)$ of $\exists x\, \varphi(x)$ by individuals of M one of the following properties holds:

- *the set of the individuals i of M such that $M \models \varphi'(i)$, is infinite,*
- *for all finite sub-set $\{\psi_1(u), .., \psi_n(u)\}$ of elements of $\Psi(u)$, the set of the individuals i of M such that $M \models \varphi'(i) \wedge \bigwedge_{j\in\{1,\dots,n\}} \neg\psi_j(i)$ is infinite.*

We write $T \models \exists_{o\,\infty}^{\Psi(u)} x\, \varphi(x)$, if for every model M of T we have $M \models \exists_{o\,\infty}^{\Psi(u)} x\, \varphi(x)$.

This quantifier is more general than the infinite quantifier and does not restrict the model to be infinite. In the case where $\Psi(u) = \{false\}$, the form (3.1) means that if $M \models \exists x\, \varphi(x)$ then M contains an infinity of individuals i such that $M \models \varphi(i)$.

Property 3.1.0.4 *Let J be a finite possibly empty set. If $T \models \exists_{o\,\infty}^{\Psi(u)} x\, \varphi(x)$ and if for each φ_j, one at least of the following properties holds:*

- *$T \models \exists ? x\, \varphi_j$,*
- *there exists $\psi_j(u) \in \Psi(u)$ such that $T \models \forall x\, \varphi_j \rightarrow \psi_j(x)$,*

then

$$T \models (\exists x\, \varphi(x) \wedge \textstyle\bigwedge_{j\in J} \neg\varphi_j) \leftrightarrow (\exists x\, \varphi(x)).$$

Proof. Let $\exists x\, \varphi'(x)$ be an instantiation of $\exists x\, \varphi(x)$ by individuals of M. Let us show that if the conditions of this property hold, then

$$M \models (\exists x\, \varphi'(x) \wedge \textstyle\bigwedge_{j\in J} \neg\varphi_j(x)) \leftrightarrow (\exists x\, \varphi'(x)). \tag{3.2}$$

Let J' be the set of the $j \in J$ such that $M \models \exists ? x\, \varphi_j(x)$ and let m be its cardinality. Since for all $j \in J'$, $M \models \exists ? x\, \varphi'_j(x)$, then it is enough that M contains at least $m+1$ individuals, to warrant the existence of an individual $i \in M$ such that

$$M \models \bigwedge_{j\in J'} \neg\varphi'_j(i). \tag{3.3}$$

On the other hand, since $T \models \exists_{o\,\infty}^{\Psi(u)} x\, \varphi(x)$ and according to Definition 3.1.0.3 of the zero-infinite quantifier, two cases arise:

(1) Either, $M \models \neg(\exists x\, \varphi'(x))$, thus $M \models \neg(\exists \bar{x}\, \varphi'(x) \wedge \bigwedge_{j\in J} \neg\varphi_j(x))$ and thus the equivalence (3.2) holds in M.

(2) Or, for every finite sub-set $\{\psi_1(u), ..., \psi_n(u)\}$ of $\Psi(u)$, the set of the individuals i of M such that $M \models \varphi'(i) \wedge \bigwedge_{j=1}^{n} \neg\psi_j(i)$ is infinite. Thus, since for all $j \in J - J'$ we have $M \models \forall x\, \varphi_j(x) \rightarrow \psi_j(x)$, then there exists an infinite set ξ of individuals i of M such that $M \models \varphi'(i) \wedge \bigwedge_{j\in J-J'} \neg\varphi_j(i)$. Since ξ is infinite, then it contains at least $m+1$ individuals and thus according to (3.3), there exists at least an individual $i \in \xi$ such that $M \models \varphi'(i) \wedge (\bigwedge_{j\in J-J'} \neg\varphi'_j(i)) \wedge (\bigwedge_{k\in J'} \neg\varphi'_k(i))$ and thus such that

$$M \models \exists x\, \varphi'(x) \wedge \bigwedge_{j\in J} \neg\varphi'_j(x).$$

Since $M \models \exists x\, \varphi'(x) \wedge \bigwedge_{j\in J} \neg\varphi_j(x)$, then $M \models \exists x\, \varphi'(x)$ and thus the equivalence (3.2) holds in M. □

Property 3.1.0.5 *If $T \models \exists_{\infty}^{\Psi(u)} x\, \varphi(x)$ then $T \models \exists_{o\,\infty}^{\Psi(u)} x\, \varphi(x)$.*

Let us recall in this section some properties of the vectorial quantifiers. These properties are proved in Chapter 2. We will handle them all long this chapter.

Property 3.1.0.6 *If $T \models \exists ?\bar{y}\phi$ and if all the variables of $\bar{y}$ has no free occurrences in φ then*

$$T \models (\exists \bar{x}\, \varphi \wedge \neg(\exists \bar{y}\, \phi \wedge \psi)) \leftrightarrow \begin{bmatrix} (\exists \bar{x}\, \varphi \wedge \neg(\exists \bar{y}\, \phi)) \\ \vee \\ (\exists \bar{x}\bar{y}\, \varphi \wedge \phi \wedge \neg\psi) \end{bmatrix}.$$

Corollary 3.1.0.7 *If $T \models \exists ?\bar{x}\, \varphi$ then*

$$T \models (\exists \bar{x}\, \varphi \wedge \bigwedge_{i\in I} \neg\phi_i) \leftrightarrow ((\exists \bar{x}\varphi) \wedge \bigwedge_{i\in I} \neg(\exists \bar{x}\, \varphi \wedge \phi_i)).$$

Corollary 3.1.0.8 *If $T \models \psi \rightarrow (\exists! \bar{x}\, \varphi)$ then*

$$T \models (\psi \wedge (\exists \bar{x}\, \varphi \wedge \bigwedge_{i\in I} \neg\phi_i)) \leftrightarrow (\psi \wedge \bigwedge_{i\in I} \neg(\exists \bar{x}\, \varphi \wedge \phi_i)).$$

3.2 Zero-infinite-decomposable theory

3.2.1 Definition

Definition 3.2.1.1 *A theory T having at least one model is called* zero-infinite-decomposable, *if there exists a set $\Psi(u)$ of formulas, having at most one free variable u, a set A of formulas closed for the conjunction, a set A' of formulas of the form $\exists \bar{x}\alpha$ with $\alpha \in A$, and a sub-set A'' of A such that*

1. *every formula of the form $\exists \bar{x}\, \alpha \wedge \psi$, with $\alpha \in A$ and ψ any formula is equivalent in T to a wnfv formula of the form*

$$\exists \bar{x}'\, \alpha' \wedge (\exists \bar{x}''\, \alpha'' \wedge (\exists \bar{x}'''\, \alpha''' \wedge \psi)),$$

with $\exists \bar{x}'\, \alpha' \in A'$, $\alpha'' \in A''$, $\alpha''' \in A$ and $T \models \forall \bar{x}''\alpha'' \rightarrow \exists! \bar{x}'''\alpha'''$,

2. *if* $\exists\bar{x}'\alpha' \in A'$*, then* $T \models \exists?\bar{x}'\,\alpha'$ *and for every free variable* y *in* $\exists\bar{x}'\alpha'$*, at least one of the following properties holds*
   - $T \models \exists?y\bar{x}'\,\alpha'$,
   - *there exists* $\psi(u) \in \Psi(u)$ *such that* $T \models \forall y\,(\exists\bar{x}'\,\alpha') \rightarrow \psi(y)$,
3. *if* $\alpha'' \in A''$ *then*
   - *the formula* $\neg\alpha''$ *is equivalent in* T *to a wnfv formula of the form* $\bigvee_{i\in I}\alpha_i$ *with* $\alpha_i \in A$,
   - *for all* x''*, the formula* $\exists x''\alpha''$ *is equivalent in* T *to a wnfv formula which belongs to* A'',
   - *for every variable* x'', $T \models \exists_{o\,\infty}^{\Psi(u)} x''\,\alpha''$,
4. *every conjunction of flat formulas is equivalent in* T *to a wnfv disjunction of elements of* A,
5. *if the formula* $\exists\bar{x}'\alpha' \wedge \alpha''$ *with* $\exists\bar{x}'\alpha' \in A'$ *and* $\alpha'' \in A''$ *has no free variables, then* $\bar{x} = \varepsilon$*, and* α' *and* α'' *belong to* $\{true, false\}$.

Note that the decomposition expressed in this definition is similar to the one defined in the infinite decomposable theories by replacing the infinite quantifier by the zero-infinite quantifier. The main difference between the two classes of theories resides in the set A'' whose properties have been increased.

3.2.2 Properties

Property 3.2.2.1 *If* T *is zero-infinite-decomposable then every formula of the form* $\exists\bar{x}\alpha$*, with* $\alpha \in A$*, is equivalent in* T*, to a wnfv formula of the form* $\exists\bar{x}'\alpha' \wedge \alpha''$*, with* $\exists\bar{x}'\alpha' \in A'$ *and* $\alpha'' \in A''$.

Proof. According to the first point of Definition 3.2.1.1, the formula $\exists\bar{x}\alpha$ is equivalent in T to a wnfv formula of the form

$$\exists\bar{x}'\,\alpha' \wedge (\exists\bar{x}''\,\alpha'' \wedge (\exists\bar{x}'''\,\alpha''')), \tag{3.4}$$

with $\exists\bar{x}'\,\alpha' \in A'$, $\alpha'' \in A''$, $\alpha''' \in A$ and $T \models \forall\bar{x}''\alpha'' \rightarrow \exists!\bar{x}'''\alpha'''$. Since $T \models \forall\bar{x}''\alpha'' \rightarrow \exists!\bar{x}'''\alpha'''$ and according to Corollary 3.1.0.8 (with ϕ is the formula *false*), the formula (3.4) is equivalent in T to

$$\exists\bar{x}'\,\alpha' \wedge (\exists\bar{x}''\,\alpha''),$$

which, since $\alpha'' \in A''$ and according to the second condition of the third point of Definition 3.2.1.1, is equivalent in T to a wnfv formula of the form

$$\exists\bar{x}'\,\alpha' \wedge \alpha''_1,$$

with $\exists\bar{x}'\,\alpha' \in A'$ and $\alpha''_1 \in A''$. □

Property 3.2.2.2 *Let* I *a finite possibly empty set. If* T *is zero-infinite-decomposable then every formula of the form*

$$\exists\bar{x}\alpha \wedge \bigwedge_{i\in I}\neg(\exists\bar{y}_i\beta_i), \tag{3.5}$$

with $\alpha \in A$ *and* $\beta_i \in A$ *for all* $i \in I$*, is equivalent in* T *to a wnfv formula of the form*

$$\exists\bar{x}'\alpha' \wedge (\exists\bar{x}''\alpha'' \wedge \bigwedge_{j\in J}\neg(\exists\bar{y}'_j\,\beta'_j \wedge \beta''_j)),$$

with $\exists\bar{x}'\alpha' \in A'$, $\alpha'' \in A''$*,* J *a finite possibly empty set with* $Card(I) = Card(J)$ *and for all* $j \in J$ *we have* $\exists\bar{y}'_j\beta'_j \in A'$ *and* $\beta''_j \in A''$.

Proof. According to the third point of Definition 3.2.1.1 (with $\psi = \bigwedge_{i\in I} \neg(\exists \bar{y}_i \beta_i)$), the formula (3.5) is equivalent in T to a wnfv formula of the form

$$\exists \bar{x}'\, \alpha' \wedge (\exists \bar{x}''\, \alpha'' \wedge (\exists \bar{x}'''\, \alpha''' \wedge \bigwedge_{j\in J} \neg(\exists \bar{y}_j \beta_j))), \tag{3.6}$$

with $\exists \bar{x}'\, \alpha' \in A'$, $\alpha'' \in A''$, $\alpha''' \in A$, $\beta_j \in A$ for all $j \in J$, $T \models \forall \bar{x}'' \alpha'' \rightarrow \exists! \bar{x}''' \alpha'''$ and $Card(I) = Card(J)$. Since $T \models \forall \bar{x}'' \alpha'' \rightarrow \exists! \bar{x}''' \alpha'''$ and according to Corollary 3.1.0.8, the formula (3.6) is equivalent in T to the wnfv formula

$$\exists \bar{x}'\, \alpha' \wedge (\exists \bar{x}''\, \alpha'' \wedge \bigwedge_{j\in J} \neg(\exists \bar{x}'''\, (\alpha''' \wedge \exists \bar{y}_j \beta_j))).$$

By lifting the quantifications $\exists \bar{y}_j$ after having possibly renamed some variables which occur in the $\bar{y}_j$, the preceding formula is equivalent in T to

$$\exists \bar{x}'\, \alpha' \wedge (\exists \bar{x}''\, \alpha'' \wedge \bigwedge_{j\in J} \neg(\exists \bar{x}''' \bar{y}_j\, \alpha''' \wedge \beta_j)),$$

which, since A is closed for the conjunction, is equivalent in T to a wnfv formula of the form

$$\exists \bar{x}'\, \alpha' \wedge (\exists \bar{x}''\, \alpha'' \wedge \bigwedge_{k\in K} \neg(\exists \bar{y}_k \beta_k)),$$

with $\exists \bar{x}'\, \alpha' \in A'$, $\alpha'' \in A''$, $\beta_k \in A$ for all $k \in K$ and $Card(K) = Card(J) = Card(I)$. According to Property 3.2.2.1, the preceding formula is equivalent in T to a wnfv formula of the form

$$\exists \bar{x}'\, \alpha' \wedge (\exists \bar{x}''\, \alpha'' \wedge \bigwedge_{\ell\in L} \neg(\exists \bar{y}'_\ell \beta'_\ell \wedge \beta''_\ell)),$$

with $\exists \bar{x}'\, \alpha' \in A'$, $\alpha'' \in A''$, for all $\ell \in L$ we have $\exists \bar{y}'_\ell \beta'_\ell \in A'$ and $\beta''_\ell \in A''$ with $Card(L) = Card(K) = Card(J) = Card(I)$. □

Corollary 3.2.2.3 *Let I be a finite possibly empty set. If T is zero-infinite-decomposable then every formula of the form*

$$\exists \bar{x}\, \alpha \wedge \bigwedge_{i\in I} \neg(\exists \bar{y}_i \beta_i), \tag{3.7}$$

with $\alpha \in A$ and $\beta_i \in A$ for all $i \in I$, is equivalent in T to a wnfv disjunction of formulas of the form

$$\exists \bar{x}'\, \alpha' \wedge (\exists \bar{x}''\, \alpha'' \wedge \bigwedge_{j\in J} \neg(\exists \bar{y}'_j\, \beta'_j)),$$

with $\exists \bar{x}' \alpha' \in A'$, $\alpha'' \in A''$, J a finite possibly empty set and for all $j \in J$ we have $\exists \bar{y}'_j\, \beta'_j \in A'$.

Proof. If I is empty then the corollary holds according to Property 3.2.2.2. Else, suppose that $I = \{1, 2, ..., n\}$ and $n \neq 0$. According to Property 3.2.2.2, the formula (3.7) is equivalent in T to a wnfv formula of the form

$$\exists \bar{x}'\, \alpha' \wedge (\exists \bar{x}'' \alpha'' \wedge \bigwedge_{i\in I} \neg(\exists \bar{y}'_i\, \beta'_i \wedge \beta''_i)), \tag{3.8}$$

with $\exists \bar{x}' \alpha' \in A'$, $\alpha'' \in A''$, $I = \{1, 2, ..., n\}$, and for all $i \in I$ we have $\exists \bar{y}'_i\, \beta'_i \in A'$ and $\beta''_i \in A''$. Thus, the formula (3.8) is equivalent in T to

$$\exists \bar{x}'\, \alpha' \wedge (\exists \bar{x}''(\alpha'' \wedge \bigwedge_{i\in I, i\neq n} \neg(\exists \bar{y}'_i\, \beta'_i \wedge \beta''_i)) \wedge \neg(\exists \bar{y}'_n\, \beta'_n \wedge \beta''_n)).$$

Since $\exists \bar{y}'_n \beta'_n \in A'$, then according to the second point of Definition 3.2.1.1 we have $T \models \exists ? \bar{y}'_n \beta'_n$. Thus, according to Corollary 3.1.0.6 (with $\varphi = \alpha'' \wedge \bigwedge_{i \in I, i \neq n} \neg(\exists \bar{y}'_i \beta'_i \wedge \beta''_i)$), the preceding formula is equivalent in T to

$$T \models \begin{bmatrix} (\exists \bar{x}'\, \alpha' \wedge (\exists \bar{x}''(\alpha'' \wedge \bigwedge_{i \in I, i \neq n} \neg(\exists \bar{y}'_i\, \beta'_i \wedge \beta''_i)) \wedge \neg(\exists \bar{y}'_n\, \beta'_n))) \\ \vee \\ (\exists \bar{x}' \alpha' \wedge (\exists \bar{x}'' \bar{y}_n\, (\alpha'' \wedge \bigwedge_{i \in I, i \neq n} \neg(\exists \bar{y}'_i\, \beta'_i \wedge \beta''_i)) \wedge \beta'_n \wedge \neg \beta''_n)) \end{bmatrix},$$

which according to the first condition of the third point of Definition 3.2.1.1 is equivalent in T to

$$T \models \begin{bmatrix} (\exists \bar{x}'\, \alpha' \wedge (\exists \bar{x}''(\alpha'' \wedge \bigwedge_{i \in I, i \neq n} \neg(\exists \bar{y}'_i\, \beta'_i \wedge \beta''_i)) \wedge \neg(\exists \bar{y}'_n\, \beta'_n))) \\ \vee \\ (\exists \bar{x}' \alpha' \wedge (\exists \bar{x}'' \bar{y}_n\, (\alpha'' \wedge \bigwedge_{i \in I, i \neq n} \neg(\exists \bar{y}'_i\, \beta'_i \wedge \beta''_i)) \wedge \beta'_n \wedge (\bigvee_{j \in J_n} \beta_{nj}))) \end{bmatrix},$$

with $T \models (\neg \beta''_n) \leftrightarrow (\bigvee_{j \in J_n} \beta_{nj})$ and $\beta_{nj} \in A$ for all $j \in J_n$. After having distributed the $\wedge$ on the $\vee$ and the $\exists$ on the $\vee$, the preceding formula is equivalent in T to

$$T \models \begin{bmatrix} (\exists \bar{x}'\, \alpha' \wedge (\exists \bar{x}''\, \alpha'' \wedge (\bigwedge_{i \in I, i \neq n} \neg(\exists \bar{y}'_i\, \beta'_i \wedge \beta''_i)) \wedge \neg(\exists \bar{y}'_n\, \beta'_n))) \\ \vee \bigvee_{j \in J_n} \\ (\exists \bar{x}' \alpha' \wedge (\exists \bar{x}'' \bar{y}_n\, \alpha'' \wedge \beta'_n \wedge \beta_{nj} \wedge \bigwedge_{i \in I, i \neq n} \neg(\exists \bar{y}'_i\, \beta'_i \wedge \beta''_i))) \end{bmatrix},$$

which by lifting the quantification $\exists \bar{x}'' \bar{y}_n$ and by renaming possibly some variables which occur in $\bar{x}'' \bar{y}_n$ is equivalent in T to

$$T \models \begin{bmatrix} (\exists \bar{x}'\, \alpha' \wedge (\exists \bar{x}''\, \alpha'' \wedge (\bigwedge_{i \in I, i \neq n} \neg(\exists \bar{y}'_i\, \beta'_i \wedge \beta''_i)) \wedge \neg(\exists \bar{y}'_n\, \beta'_n))) \\ \vee \bigvee_{j \in J_n} \\ (\exists \bar{x}' \exists \bar{x}'' \bar{y}_n\, \alpha' \wedge \alpha'' \wedge \beta'_n \wedge \beta_{nj} \wedge \bigwedge_{i \in I, i \neq n} \neg(\exists \bar{y}'_i\, \beta'_i \wedge \beta''_i)) \end{bmatrix},$$

which according to Property 3.2.2.2 (because A is closed for the conjunction and A'' is a sub-set of A), is equivalent in T to a wnfv formula of the form

$$T \models \begin{bmatrix} (\exists \bar{x}'\, \alpha' \wedge (\exists \bar{x}''\, \alpha'' \wedge (\bigwedge_{i \in I, i \neq n} \neg(\exists \bar{y}'_i\, \beta'_i \wedge \beta''_i)) \wedge \neg(\exists \bar{y}'_n\, \beta'_n))) \\ \vee \bigvee_{j \in J_n} \\ (\exists \bar{x}'_j\, \alpha'_j \wedge \alpha''_j \wedge \bigwedge_{i \in I, i \neq n} \neg(\exists \bar{z}'_{ij}\, \delta'_{ij} \wedge \delta''_{ij})) \end{bmatrix}, \tag{3.9}$$

with $\exists \bar{x}' \alpha' \in A'$, $\alpha'' \in A''$, $I = \{1, 2, ..., n\}$, $\exists \bar{y}'_n \beta'_n \in A'$, for all $i \in I$ with $i \neq n$ we have $\exists \bar{y}'_i \beta'_i \in A'$ and $\beta''_i \in A''$ and for all $j \in J_n$ we have $\exists \bar{x}'_j \alpha'_j \in A'$, $\alpha''_j \in A''$, $\exists \bar{z}'_{ij} \delta'_{ij} \in A'$ and $\delta''_{ij} \in A''$.

Thus, starting from the formula (3.8) which has $car(I) = n$ sub-formulas of the form

$$\neg(\exists \bar{y}'_i \beta'_i \wedge \beta''_i), \tag{3.10}$$

with $\exists \bar{y}'_i \beta'_i \in A'$ and $\beta''_i \in A''$, we get a wnfv disjunction of formulas each one containing $card(I) - 1 = n - 1$ sub-formulas of the form (3.10). Thus, we have

(1) By repeating another time the preceding steps on the first formula of (3.9) of the form

$$\exists \bar{x}'\, \alpha' \wedge (\exists \bar{x}''\, \alpha'' \wedge (\bigwedge_{i \in I, i \neq n} \neg(\exists \bar{y}'_i\, \beta'_i \wedge \beta''_i)) \wedge \neg(\exists \bar{y}'_n\, \beta'_n)),$$

we get a wnfv formula equivalent in T of the form

$$T \models \begin{bmatrix} (\exists \bar{x}'\, \alpha' \wedge (\exists \bar{x}''\, \alpha'' \wedge (\bigwedge_{i \in I, i \neq n, i \neq n-1} \neg(\exists \bar{y}'_i\, \beta'_i \wedge \beta''_i)) \wedge \neg(\exists \bar{y}'_{n-1}\, \beta'_{n-1}) \wedge \neg(\exists \bar{y}'_n\, \beta'_n))) \\ \vee \bigvee_{j \in J_{n-1}} \\ (\exists \bar{x}'_j\, \alpha'_j \wedge \alpha''_j \wedge \bigwedge_{i \in I, i \neq n} \neg(\exists \bar{z}'_{ij}\, \delta'_{ij} \wedge \delta''_{ij})) \end{bmatrix}, \tag{3.11}$$

with $\exists\bar{x}'\alpha' \in A'$, $\alpha'' \in A''$, $I = \{1, 2, ..., n\}$, $\exists\bar{y}'_{n-1}\beta'_{n-1} \in A'$, $\exists\bar{y}'_n\beta'_n \in A'$, for all $i \in I$ with $i \neq n$ and $i \neq n-1$ we have $\exists\bar{y}'_i\beta'_i \in A'$ and $\beta''_i \in A''$ and for all $j \in J_{n-1}$ we have $\exists\bar{x}'_j\alpha'_j \in A'$, $\alpha''_j \in A''$, $\exists\bar{z}'_{ij}\delta'_{ij} \in A'$ and $\delta''_{ij} \in A''$.

(2) By repeating the preceding steps in each sub-formula of (3.9) or (3.11) of the form

$$\exists\bar{x}'_j\, \alpha'_j \wedge \alpha''_j \wedge \bigwedge_{i\in I, i\neq n} \neg(\exists\bar{z}'_{ij}\, \delta'_{ij} \wedge \delta''_{ij}),$$

we get a wnfv formula equivalent in T of the form

$$T \models \begin{bmatrix} (\exists\bar{x}'\, \alpha' \wedge (\exists\bar{x}''\, \alpha'' \wedge (\bigwedge_{i\in I, i\neq n, i\neq n-1} \neg(\exists\bar{y}'_i\, \beta'_i \wedge \beta''_i)) \wedge \neg(\exists\bar{y}'_{n-1}\, \beta'_{n-1}))) \\ \vee \bigvee_{j\in J_{n-1}} \\ (\exists\bar{x}'_j\, \alpha'_j \wedge \alpha''_j \wedge \bigwedge_{i\in I, i\neq n, i\neq n-1} \neg(\exists\bar{z}'_{ij}\, \delta'_{ij} \wedge \delta''_{ij})) \end{bmatrix},$$

with $\exists\bar{x}'\alpha' \in A'$, $\alpha'' \in A''$, $I = \{1, 2, ..., n\}$, $\exists\bar{y}'_{n-1}\beta'_{n-1} \in A'$, for all $i \in I$ with $i \neq n$ and $i \neq n-1$ we have $\exists\bar{y}'_i\beta'_i \in A'$ and $\beta''_i \in A''$ and for all $j \in J_{n-1}$ we have $\exists\bar{x}'_j\alpha'_j \in A'$, $\alpha''_j \in A''$, $\exists\bar{z}'_{ij}\delta'_{ij} \in A'$ and $\delta''_{ij} \in A''$.

From (1) and (2) we deduce that it is enough to apply the preceding steps a finite numbers of time on each disjunction by saving the formulas of the form $\neg(\exists\bar{y}'_i\beta'_i)$, to eliminate all sub-formulas of the form $\neg(\exists\bar{y}'_i\beta'_i \wedge \beta''_i)$. At the end we get a disjunction of formulas of the form

$$\exists\bar{x}'\, \alpha' \wedge (\exists\bar{x}''\, \alpha'' \wedge \bigwedge_{j\in J} \neg(\exists\bar{y}'_j\, \beta'_j)),$$

with $\exists\bar{x}'\alpha' \in A'$, $\alpha'' \in A''$, J a finite possibly empty set and for all $j \in J$ we have $\exists\bar{y}'_j\, \beta'_j \in A'$. □

3.2.3 Completeness

Theorem 3.2.3.1 *If T is zero-infinite-decomposable then T is complete.*

Proof. Let T be a zero-infinite-decomposable theory which satisfies the five conditions of Definition 2.2.1.1. Let us show that T is complete using Property 1.2.3.1 by taking formulas of the form $\exists\bar{x}\, \alpha$, with $\alpha \in A$ as basic formulas. Note that according to Definition 3.2.1.1, the formulas of A' are of the form $\exists\bar{x}\alpha$ with $\alpha \in A$ and A'' is a sub-set of A.

Let us show that the first condition of Property 1.2.3.1 holds, i.e. every flat formula is equivalent in T to a wnfv boolean combination of basic formulas. If φ is a flat formula, then according to the fourth point of Definition 3.2.1.1, φ is equivalent in T to a disjunction of elements of A, thus to a disjunction of formulas of the form $\exists\varepsilon\, \alpha$ with $\alpha \in A$, which is a boolean combination of basic formulas.

Let us show that the second condition of property 1.2.3.1 holds, i.e. every basic formula without free variables is equivalent, either to *true*, or to *false* in T. Let $\exists\bar{x}\, \alpha$ with $\alpha \in A$ be a basic formula without free variables. According to Property 3.2.2.1, this formula is equivalent in T to a wnfv formula of the form $\exists\bar{x}'\, \alpha' \wedge \alpha''$ with $\exists\bar{x}'\, \alpha' \in A'$ and $\alpha'' \in A''$. According to the fifth point of Definition 3.2.1.1, we have $\bar{x} = \varepsilon$, $\alpha' \in \{true, false\}$ and $\alpha'' \in \{true, false\}$. Since T has at least one model then either $T \models \varphi$ or $T \models \neg\varphi$.

Let us show that the third condition of Property 1.2.3.1 holds, i.e. every formula of the form

$$\exists x\, (\bigwedge_{i\in I}(\exists\bar{x}_i\, \alpha_i)) \wedge (\bigwedge_{j\in J} \neg(\exists\bar{y}_j\, \beta_j)), \tag{3.12}$$

with $\alpha_i \in A$ for all $i \in I$ and $\beta_j \in A$ for all $j \in J$, is equivalent in T to a wnfv boolean combination of basic formulas, i.e. a wnfv boolean combination of formulas of the form $\exists\bar{x}\alpha$ with

$\alpha \in A$. By lifting the quantifications $\exists \bar{x}_i$ after having possibly renamed some variables which occur in each $\bar{x}_i$, the formula (3.12) is equivalent in T to wnfv formula of the form

$$\exists \bar{x}\,(\bigwedge_{i\in I}\alpha_i) \wedge \bigwedge_{j\in J}\neg(\exists \bar{y}_j\,\beta_j),$$

with $\alpha_i \in A$ for all $i \in I$ and $\beta_j \in A$ for all $j \in J$. According to Definition 3.2.1.1, the set A is closed for the conjunction. Thus, the preceding formula is equivalent in T to a wnfv formula of the form

$$\exists \bar{x}\,\alpha \wedge \bigwedge_{j\in J}\neg(\exists \bar{y}_j\,\beta_j),$$

with $\alpha \in A$ and $\beta_j \in A$ for all $j \in J$. According to Corollary 3.2.2.3, the preceding formula is equivalent in T to a wnfv disjunction of formulas of the form

$$\exists \bar{x}'\,\alpha' \wedge (\exists \bar{x}''\alpha'' \wedge \bigwedge_{i\in I}\neg(\exists \bar{y}'_i\beta'_i)). \tag{3.13}$$

with $\exists \bar{x}'\,\alpha' \in A'$, $\alpha'' \in A''$ and $\exists \bar{y}'_i\beta'_i \in A'$ for all $i \in I$. Let us show that each formula of this disjunction is equivalent in T to a wnfv boolean combination of basic formulas. Let φ be a formula of the form (3.13). Let us denote by I_1 the set of the $i \in I$ such that x''_n has no occurrences in $\exists \bar{y}'_i\beta'_i$. The formula φ is equivalent in T to

$$\exists \bar{x}'\alpha' \wedge (\exists x''_1 ... \exists x''_{n-1}\begin{bmatrix}(\bigwedge_{i\in I_1}\neg(\exists \bar{y}'_i\beta'_i))\wedge \\ (\exists x''_n\,\alpha'' \wedge \bigwedge_{i\in I-I_1}\neg(\exists \bar{y}'_i\beta'_i))\end{bmatrix}).$$

Since $\alpha'' \in A''$ and $\exists \bar{y}'_i\beta'_i \in A'$ for all $i \in I$ and according to Property 3.1.0.4 and the points 2 and 3 of Definition 3.2.1.1, the preceding formula is equivalent in T, to

$$\exists \bar{x}'\alpha' \wedge (\exists x''_1 ... \exists x''_{n-1}\,(\bigwedge_{i\in I_1}\neg(\exists \bar{y}'_i\beta'_i)) \wedge (\exists x''_n\alpha'')),$$

which, since $\alpha'' \in A''$ and according to the second point of the third condition Definition 3.2.1.1, is equivalent in T to a wnfv formula of the form

$$\exists \bar{x}'\alpha' \wedge (\exists x''_1 ... \exists x''_{n-1}\,(\bigwedge_{i\in I_1}\neg(\exists \bar{y}'_i\beta'_i)) \wedge \alpha''_n),$$

with $\exists \bar{x}'\alpha' \in A'$, $\alpha''_n \in A''$ and $\exists \bar{y}'_i\beta'_i \in A'$ for all $i \in I_1$, i.e. to

$$\exists \bar{x}'\alpha' \wedge (\exists x''_1 ... \exists x''_{n-1}\,\alpha''_n \wedge \bigwedge_{i\in I_1}\neg(\exists \bar{y}'_i\beta'_i)).$$

By repeating the three preceding steps $n-1$ times and by denoting by I_k the set of the $i \in I_{k-1}$ such that $x''_{(n-k+1)}$ has no occurrences in $\exists \bar{y}'_i\beta'_i$, we get a wnfv formula equivalent in T, of the form

$$\exists \bar{x}'\alpha' \wedge \alpha''_1 \wedge \bigwedge_{i\in I_n}\neg(\exists \bar{y}'_i\beta'_i),$$

with $\exists \bar{x}'\,\alpha' \in A'$, $\alpha''_1 \in A''$ and $\exists \bar{y}'_i\beta'_i \in A'$ for all $i \in I_n$. Since $\exists \bar{x}'\alpha' \in A'$, then according to the second point of Definition 3.2.1.1, we have $T \models \exists ? \bar{x}'\alpha'$, thus $T \models \exists ? \bar{x}'\alpha' \wedge \alpha'_1$. According to Corollary 3.1.0.7, the preceding formula is equivalent in T to

$$(\exists \bar{x}'\alpha' \wedge \alpha''_1) \wedge \bigwedge_{i\in I_n}\neg(\exists \bar{x}'\,\alpha' \wedge \alpha''_1 \wedge \exists \bar{y}'_i\beta'_i),$$

which by lifting the quantifications $\exists \bar{y}'_i$ and by renaming some variables which occur in each $\bar{y}'_i$, is equivalent in T to a wnfv of the form

$$(\exists \bar{x}'\alpha' \wedge \alpha''_1) \wedge \bigwedge_{i\in I_n}\neg(\exists \bar{x}'\bar{y}'_i\,\alpha' \wedge \alpha''_1 \wedge \beta'_i),$$

with α', α''_1, and all the β'_i element of A, and $\exists\bar{x}\alpha' \in A'$. Since the formulas α', α''_1, β'_i belong to A and since A is closed for the conjunction, then the preceding formula is equivalent in T to a wnfv formula of the form

$$(\exists\bar{x}\alpha) \wedge \bigwedge_{i\in I_n} \neg(\exists\bar{y}_i\, \beta_i),$$

with $\alpha \in A$ and $\beta_i \in A$ for all $i \in I$. This formula is a boolean combination of formulas of the form $\exists\bar{x}\alpha$ with $\alpha \in A$, i.e. a boolean combination of basic formulas. Thus, the third condition of Property 1.2.3.1 holds.

Since T satisfies the three conditions of Property 1.2.3.1, then T is a complete theory. □

According to Theorem 3.2.3.1 and Corollary 1.2.3.2, we have the following corollary:

Corollary 3.2.3.2 *If T is zero-infinite-decomposable and if for every formula of the form $\exists\bar{x}'\alpha' \wedge \alpha''$ which belongs to A' we have $\bar{x}' = \varepsilon$ and $\alpha' \wedge \alpha'' \in AT$, then T accepts a full elimination of quantifiers.*

Proof. Let T be a zero-infinite-decomposable theory such that for each formula of the form $\exists\bar{x}'\alpha' \wedge \alpha''$ which belongs to A' we have $\bar{x}' = \varepsilon$ and $\alpha' \wedge \alpha'' \in AT$. Let φ be formula. In the proof of Theorem 3.2.3.1, we have shown that T satisfies the three conditions of Property 1.2.3.1 using formulas of the form $\exists\bar{x}\alpha$ with $\alpha \in A$ as basic formulas. Thus, according to Corollary 1.2.3.2, the formula φ is equivalent in T to a wnfv boolean combination of basic formulas, i.e. a wnfv boolean combination of formulas of the form $\exists\bar{x}\alpha$ with $\alpha \in A$. According to Property 3.2.2.1, each one of these basic formulas is equivalent in T to a wnfv formula of the form $\exists\bar{x}'\alpha' \wedge \alpha''$ with $\exists\bar{x}'\,\alpha' \in A'$ and $\alpha'' \in A''$. Since $\bar{x}' = \varepsilon$ and since $\alpha' \wedge \alpha'' \in AT$, then the formula φ is equivalent in T to a wnfv boolean combination ϕ of conjunctions of atomic formulas. According to the definition of the atomic formulas (Chapter 1), it is clear that ϕ does not contain quantifiers. □

Let us now present a property that links the infinite-decomposable theories to the zero-infinite-decomposable theories. According to Property 3.1.0.5 and Property 2.1.2.3, we have:

Property 3.2.3.3 *An infinite-decomposable theory T is zero-infinite decomposable if for each formula of the form $\exists\bar{x}''\,\alpha'' \in A''$, the formula $\neg\alpha''$ is equivalent in T to a disjunction of elements of A.*

It is obvious that the sets A and A'' cited in this property are those that appear in the definition of the infinite-decomposable theory and not those of the definition of the zero-infinite-decomposable theories. The theories Eq, Ra and $\mathcal{T}$ presented in Chapter 2, are zero-infinite-decomposable. In fact, we have shown their infinite-decomposability using a set A'' containing formulas of the form $\exists\bar{x}''\, true$. Thus, since $\neg true$ is equivalent to the formula *false* in all these theories and since the formula *false* belongs to A, then each one of these theories are zero-infinite-decomposable.

3.2.4 Fundamental example

Let F be an empty set of function symbols and R a set of relation symbols containing only the binary relation symbol $<$. If t_1 and t_2 are terms, then we write $t_1 < t_2$ for $<(t_1, t_2)$. Let T_{ord} be the theory of the linear dense order relation without endpoints, whose signature is $S = F \cup R$ and whose set of axioms is the set of the following propositions:

1 $\forall x\, \neg x < x$,
2 $\forall x \forall y \forall z\, (x < y \wedge y < z) \rightarrow x < z$,
3 $\forall x \forall y\, x < y \vee x = y \vee y < x$,
4 $\forall x \forall y\, x < y \rightarrow (\exists z\, x < z \wedge z < y)$,
5 $\forall x\, \exists y\, x < y$,
6 $\forall x\, \exists y\, y < x$.

Since $F = \emptyset$ then all the equations and relations are flat. Let us introduce three properties that will help us to show the zero-infinite-decomposability of this theory. These properties are well known and deduced from the preceding axiomatization. The first one shows the full elimination of quantifiers of Fourier. The second one shows the behavior of the negation with the relation $<$ and the last one introduces the notion of zero-infinite in every model of T_{ord} due to the fact that T_{ord} is dense and without endpoints.

Property 3.2.4.1 *Let I and J be finite possibly empty sets. We have*

$$T_{ord} \models (\exists x\, (\bigwedge_{i\in I} x < y_i) \wedge (\bigwedge_{j\in J} z_j < x)) \leftrightarrow \bigwedge_{i\in I} \bigwedge_{j\in J} (z_j < y_i).$$

Property 3.2.4.2

$$T_{ord} \models \forall xy\, \neg(x < y) \leftrightarrow ((x = y) \vee (y < x)).$$

Property 3.2.4.3 *Let M be a model of T_{ord}. Let J and K be two finite possibly empty sets of individuals of M and let $\varphi(x)$ be the following M-formula:*

$$(\bigwedge_{j\in J} j < x) \wedge (\bigwedge_{k\in K} x < k).$$

The set of the individuals i of M such that $M \models \varphi(i)$ is empty or infinite

Suppose that the variables of V are ordered by a linear dense order relation without endpoints denoted by $\succ$.

Definition 3.2.4.4 *A conjunction α of flat formulas is called $(\succ)$-solved in T_{ord} if*

- *all the equations of α are of the form $x = y$ with $x \succ y$,*
- *all the left hand sides of the equations of α are distinct and have one and only occurrence in α,*
- *α does not contain sub-formulas of the form false or*

$$x_0 < x_1 \wedge x_1 < x_2 \wedge ... \wedge x_{n-1} < x_n \wedge x_n < x_0.$$

Example 3.2.4.5 *Let x, y z and w be variables such that $x \succ y \succ z \succ w$. The formula $x = y \wedge z < x$ is not $(\succ)$-solved because x occurs in the left hand side of the equation $x = y$ and occurs also in the relation $z < x$. The formula $x = y \wedge y < z \wedge z < w \wedge w < y$ is not $(\succ)$-solved because the last condition of Definition 3.2.4.4 does not hold. The formula $x = z \wedge y = z \wedge z < w$ is $(\succ)$-solved.*

Property 3.2.4.6 *Every conjunction of flat formulas is equivalent in T_{ord}, either to false, or to a $(\succ)$-solved wnfv conjunction of flat formulas.*

Proof. Let us introduce the set of the following rewriting rules:

(1)	$x = x$	$\Longrightarrow$	$true$,
(2)	$y = x$	$\Longrightarrow$	$x = y$,
(3)	$x = y \wedge x = z$	$\Longrightarrow$	$x = y \wedge z = y$,
(4)	$x = y \wedge z = x$	$\Longrightarrow$	$x = y \wedge z = y$,
(5)	$x = y \wedge x < z$	$\Longrightarrow$	$x = y \wedge y < z$,
(6)	$x = y \wedge z < x$	$\Longrightarrow$	$x = y \wedge z < y$,
(7)	$false \wedge \alpha$	$\Longrightarrow$	$false$,
(8)	$x_0 < x_1 \wedge ... \wedge x_{n-1} < x_n \wedge x_n < x_0$	$\Longrightarrow$	$false$,

The rules (2)...(6) are applied only if $x \succ y$. This condition prevents infinite loops. In the rule (8), n is a possibly nul natural numbers[11]. It is clear that every repeated application of the preceding rewriting rules on a conjunction of flat formulas is terminating, and produces either *false* or a wnfv ($\succ$)-solved conjunction of flat equations equivalent in T_{ord}.

Property 3.2.4.7 *Let α be a ($\succ$)-solved conjunction of equations and $\bar{x}$ the vector the left hand sides of the equations of α. Let β be a ($\succ$)-solved conjunction of relations. We have*

1. $T_{ord} \models \exists! \bar{x}\, \alpha$.
2. $T_{ord} \models \exists_{o\,\infty}^{\{false\}} x\, \beta$.
3. *for all $x \in var(\alpha)$ we have* $T_{ord} \models \exists ? x\, \alpha$.

The first point holds since all the left hand sides of the equations of α are distinct and have one and only one occurrence in α. Thus, for each model of T_{ord} and for each instantiation of the variables which occur in the right hand sides of the equations of α, there exists one and only value for the left hand sides of the equations of α. The second point is a consequence of Property 3.2.4.3 and holds since the domains of all the models of T_{ord} are infinite[12]. The third point holds since in a ($\succ$)-solved conjunction of equations there exists no formulas of the form $x = x$ (because $x \not\succ x$). Thus, using the properties of the equality, for every model of T_{ord} and for every instantiation of the variables of $var(\alpha) - \{x\}$, either there exists a unique solution for x, or there exists a contradiction in this instantiation and thus there exists no possible values for x.

Property 3.2.4.8 *The theory T_{ord} is zero-infinite-decomposable.*

Proof. Let us show that T_{ord} satisfies the conditions of Definition 3.2.1.1. The sets A, A', A'' and $\Psi(u)$ are chosen as follows:

- A is the set PL.
- A' is the set of formulas of the form $\exists \varepsilon\, \alpha'$ where α' is either the formula *false*, or a ($\succ$)-solved conjunction of flat equations,
- A'' is the set of the ($\succ$)-solved conjunctions of relations,
- $\Psi(u) = \{false\}$.

It is clear that, FL is closed for the conjunction, A' contains formulas of the form $\exists \bar{x}'\, \alpha'$ with $\alpha' \in FL$ and A'' is a sub-set of FL.

Let us show that T_{ord} satisfies the first condition of Definition 3.2.1.1. Let $\alpha \in FL$ and ψ any formula. Let $\bar{x}$ be a vector of variables. Let us choose the order $\succ$ such that the variables of $\bar{x}$ are greater than the free variables of $\exists \bar{x}\, \alpha$. According to Property 3.2.4.6 two cases arise:

Either, the formula α is equivalent to *false* in T_{ord} and thus the formula $\exists \bar{x} \alpha \wedge \psi$ is equivalent in T_{ord} to a decomposed formula of the form

$$\exists \varepsilon\, false \wedge (\exists \varepsilon\, true \wedge (\exists \varepsilon\, true \wedge \psi)).$$

[11] If $n = 0$, then the rule will be of the form $x_0 < x_0 \leftrightarrow false$.

[12] Since each model has at least one individual, then using the axioms 1,5 and 6 we create an infinity of distinct individuals in every model of T_{ord}.

Or, the formula α is equivalent in T_{ord} to a ($\succ$)-solved conjunction β of flat equations and relations. Let X_g be the set of the variables of $\bar{x}$ which occur in a left hand side of equations of β. Let X_n be the set of the variables of $\bar{x}$ which do not occur in left hand sides of equations of β. The formula $\exists\bar{x}\alpha \wedge \psi$ is equivalent in T_{ord} to a decomposed formula of the form

$$\exists\bar{x}'\,\alpha' \wedge (\exists\bar{x}''\,\alpha'' \wedge (\exists\bar{x}'''\,\alpha''' \wedge \psi)), \tag{3.14}$$

with $\bar{x}' = \varepsilon$. The formula α' contains the conjunction of the equations of β whose left hand sides do not belong to X_g, i.e. whose left hand sides are free in $\exists\bar{x}\beta$. The vector $\bar{x}''$ contains the variables of X_n. The formula α'' contains the conjunction of the relations of β. The vector $\bar{x}'''$ contains the variables of X_g. The formula α''' is the conjunction of the equations of β whose left hand sides belong to X_g. According to our construction, it is clear that $\exists\bar{x}'\alpha' \in A'$, $\alpha'' \in A''$ and $\alpha''' \in A$. Moreover, according to the first point of Property 3.2.4.7, we have $T_{ord} \models \forall\bar{x}''\alpha'' \rightarrow \exists!\bar{x}'''\alpha'''$. Let us show now that (3.14) and $\exists\bar{x}\alpha \wedge \psi$ are equivalent in T_{ord}. Let X, X', X'' and X''' be the sets of the variables of the vectors[13] of $\bar{x}$, $\bar{x}'$, $\bar{x}''$ and $\bar{x}'''$. If α is equivalent to *false* in T_{ord} then the equivalence of the decomposition is evident. Else, β is a ($\succ$)-solved conjunction of equations and relations. Thus, according to our construction we have: $X = X' \cup X'' \cup X'''$, $X' \cap X'' = \emptyset$, $X' \cap X''' = \emptyset$, $X'' \cap X''' = \emptyset$, $X' = \emptyset$, for all $x_i'' \in X''$ we have $x_i'' \notin var(\alpha')$ and for all $x_i''' \in X'''$ we have $x_i''' \notin var(\alpha' \wedge \alpha'')$. These properties come from the definition of a ($\succ$)-solved conjunction of flat formulas and the order $\succ$ which has been chosen such that the quantified variables of $\exists\bar{x}\,\alpha$ are greater than the free variables of $\exists\bar{x}\,\alpha$. On the other hand, each equation and each relation of β occurs in $\alpha' \wedge \alpha'' \wedge \alpha'''$ and each equation and each relation of $\alpha' \wedge \alpha'' \wedge \alpha'''$ occurs in β and thus $T_{ord} \models \beta \leftrightarrow (\alpha' \wedge \alpha'' \wedge \alpha''')$. We have shown that the quantifications are coherent and the equivalence $T_{ord} \models \beta \leftrightarrow \alpha' \wedge \alpha'' \wedge \alpha'''$ holds. According to Property 3.2.4.6, we have $T_{ord} \models \alpha \leftrightarrow \beta$ and thus the decomposition preserves the equivalence in T_{ord}.

Example 3.2.4.9 *Let us decompose the formula*

$$\exists xyz\, v = w \wedge z = z \wedge z = x \wedge v = y \wedge v < z.$$

Let us choose first an order $\succ$ such that $x \succ y \succ z \succ v \succ w$. Let us now transform the formula $v = w \wedge z = z \wedge z = x \wedge v = y \wedge v < z$ into a ($\succ$)-solved conjunction of flat formulas. The preceding formula is equivalent in T_{ord} to

$$\exists xyz\, v = w \wedge x = z \wedge y = w \wedge w < z.$$

This formula is equivalent in T_{ord} to a decomposed formula of the form

$$\exists\varepsilon\, v = w \wedge (\exists z\, w < z \wedge (\exists xy\, x = z \wedge y = w)).$$

The theory T_{ord} satisfies the second condition of Definition 3.2.1.1 according to the third point of Property 3.2.4.7 and using the fact that $\bar{x}' = \varepsilon$. The theory T_{ord} satisfies the first point of the third condition of Definition 3.2.1.1 according to Property 3.2.4.2 which enables us to show that every formula of the form $\neg\varphi$ with φ a ($\succ$)-solved conjunction of relations is equivalent in T_{ord} to a disjunction of relations and equations, thus to a disjunction of elements of FL. The theory T_{ord} satisfies the second point of the third condition of Definition 3.2.1.1 according to Property 3.2.4.1. The theory T_{ord} satisfies the third point of the third condition of Definition 3.2.1.1 according to the second point of Property 3.2.4.7. The theory T_{ord} satisfies the fourth

[13] Of course, if $\bar{x} = \varepsilon$ then $X = \emptyset$

condition of Definition 3.2.1.1 since $A = FL$. The theory T_{ord} satisfies the last condition of Definition 3.2.1.1 because (1) A' is of the form $\exists\varepsilon\, \alpha'$ where α' is either the formula *false*, or a $(\succ)$-solved conjunction of flat equations, (2) A'' contains $(\succ)$-solved conjunctions of relations. Thus, if $\exists\varepsilon\, \alpha' \wedge \alpha''$ has no free variables then $\alpha' \in \{true, false\}$ and $\alpha'' \in \{true\}$.

We have shown that T_{ord} satisfies all the conditions of Definition 3.2.1.1. Thus, it is zero-infinite-decomposable. □

Note that T_{ord} accepts a full elimination of quantifiers. In fact, Corollary 3.2.3.2 confirms this result since for each formula $\exists\bar{x}'\alpha' \wedge \alpha''$ with $\exists\bar{x}'\alpha' \in A'$ and $\alpha'' \in A''$ we have $\bar{x}' = \varepsilon$ and $\alpha' \wedge \alpha'' \in FL$.

3.3 A decision procedure in zero-infinite-decomposable theories

Let T be a zero-infinite-decomposable theory together with its set of function symbols F and its set of relation symbols R. The sets $\Psi(u)$, A, A' and A'' are now known and fixed.

3.3.1 Normalized Formula

Definition 3.3.1.1 *a normalized formula φ of depth $d \geq 1$ is a formula of the form*

$$\neg(\exists\bar{x}\, \alpha \wedge \bigwedge_{i\in I} \varphi_i), \tag{3.15}$$

with I finite possibly empty set, $\alpha \in FL$, all the φ_i are normalized formulas of depth d_i with $d = 1 + \max\{0, d_1, ..., d_n\}$ and all the quantified variables of φ have distinct names and different form those of the free variables.

Note that the normalized formulas defined in Chapter 2 of infinite-decomposable theories are the same than those defined here. Thus, we can use the following property[14]

Property 3.3.1.2 *every formula φ is equivalent in the empty theory[15] to a wnfv normalized formula of depth $d \geq 1$.*

3.3.2 Working formula

Definition 3.3.2.1 *a working formula φ of depth $d \geq 1$ is a formula of the form*

$$\neg(\exists\bar{x}\, \alpha \wedge \bigwedge_{i\in I} \varphi_i), \tag{3.16}$$

with I a finite possibly empty set, $\alpha \in A$, all the φ_i are working formulas of depth d_i with $d = 1 + \max\{0, d_1, ..., d_n\}$ and all the quantified variables of φ have distinct names and different from those of the free variables.

Property 3.3.2.2 *Every formula is equivalent in T to a wnfv conjunction of working formulas.*

Proof. Let φ be a formula. According to Property 3.3.1.2, φ is equivalent in T to a wnfv normalized formula ϕ. Let us show by recurrence on the depth of ϕ that ϕ is equivalent in T to a working formula. If ϕ is of depth 1, then it is of the form

$$\neg(\exists\bar{x}\alpha), \tag{3.17}$$

[14] Already proved in Property 2.3.1.3 of chapter 2.
[15] Thus in every theory.

with $\alpha \in FL$. According to the point 4 of Definition 3.2.1.1, each conjunction of flat formulas is equivalent in T to a disjunction of elements of A. Thus, there exists a wnfv disjunction $\bigvee_{j \in J} \alpha_j$ with $\alpha_j \in A$ for all $j \in J$ such that $T \models \alpha \leftrightarrow \bigvee_{j \in J} \alpha_j$. Thus, the formula (3.17) is equivalent in T to

$$\neg(\exists \bar{x} \, (\bigvee_{j \in J} \alpha_j)),$$

which is equivalent in T to

$$\neg \bigvee_{j \in J} (\exists \bar{x} \, \alpha_j),$$

i.e. to

$$\bigwedge_{j \in J} \neg(\exists \bar{x} \, \alpha_j),$$

which by renaming the quantified variables by distinct names and different from those of the free variables gives a conjunction of working formulas. Suppose now that every normalized formula of depth n is equivalent in T to a conjunction of working formulas, and let us show that every normalized formula of depth $n+1$ is equivalent in T to a conjunction of working formulas. Let ϕ be a normalized formula of depth $n+1$. The formula ϕ is of the form

$$\neg(\exists \bar{x} \, \alpha \wedge \bigwedge_{i \in I} \varphi_i), \tag{3.18}$$

where all the φ_i are normalized formulas of depth less or equal to n. According to the recurrence hypothesis, each one of these normalized formulas φ_i is equivalent in T to a conjunction of working formulas. Thus, the formula (3.18) is equivalent in T to a formula of the form

$$\neg(\exists \bar{x} \, \alpha \wedge \bigwedge_{i \in I} \varphi_i), \tag{3.19}$$

where all the φ_i are working formulas and $\alpha \in A$. According to the point 4 of Definition 3.2.1.1, every conjunction of flat formulas is equivalent in T to a disjunction of elements of A. Thus, there exists a wnfv disjunction $\bigvee_{j \in J} \alpha_j$ with $\alpha_j \in A$ for all $j \in J$ and such that $T \models \alpha \leftrightarrow \bigvee_{j \in J} \alpha_j$. Then, the formula (3.19) is equivalent in T to

$$\neg(\exists \bar{x} \, (\bigvee_{j \in J} \alpha_j) \wedge \bigwedge_{i \in I} \varphi_i), \tag{3.20}$$

which is equivalent in T to

$$\neg(\exists \bar{x} \, \bigvee_{j \in J} (\alpha_j \wedge \bigwedge_{i \in I} \varphi_i)),$$

i.e. to

$$\neg \bigvee_{j \in J} (\exists \bar{x} \, \alpha_j \wedge \bigwedge_{i \in I} \varphi_i),$$

and thus to

$$\bigwedge_{j \in J} \neg(\exists \bar{x} \, \alpha_j \wedge \bigwedge_{i \in I} \varphi_i),$$

which by renaming the quantified variables by distinct names and different from those of the free variables gives a conjunction of working formulas. □

Definition 3.3.2.3 *A* solved *formula is a formula of the form*

$$\neg(\exists \bar{x}'\, \alpha' \wedge \alpha'' \wedge \bigwedge_{i\in I} \neg(\exists \bar{y}'_i\, \beta'_i)),$$

with I a finite possibly empty set, $\exists \bar{x}'\alpha' \in A'$, $\alpha'' \in A''$, $\exists \bar{y}'_i \beta'_i \in A'$ for all $i \in I$, α' and α'' are different from the formula false and all the β'_i are different from the formulas true and false.

Property 3.3.2.4 *Let φ be a conjunction of solved formulas without free variables. The conjunction φ is either the formula true, or the formula $\bigwedge \neg true$.*

Proof. Let φ be a conjunction of solved formulas without free variables. According to Definition 3.3.2.3, φ is of the form

$$true \wedge \bigwedge_{i\in I} \neg(\exists \bar{x}'_i \alpha'_i \wedge \alpha''_i \wedge \bigwedge_{j\in J_i} \neg(\exists \bar{y}'_{ij} \beta'_{ij})) \tag{3.21}$$

with

1. I a finite possibly empty set,
2. $(\exists \bar{x}'_i \alpha'_i) \in A'$ and $\alpha''_i \in A''$ for all $i \in I$,
3. $(\exists \bar{y}'_{ij} \beta'_{ij}) \in A'$ for all $i \in I$ and all $j \in J_i$,
4. α'_i and α''_i are different from *false* for all $i \in I$,
5. β'_{ij} is different from *true* and *false* for all $i \in I$ and all $j \in J_i$.

Since these solved formulas have no free variables and since T is zero-infinite-decomposable then according to the fifth point of Definition 3.2.1.1 and the conditions 2 and 3 of (3.21) we have

- (*) for every formula $\exists \bar{x}'_i \alpha'_i \wedge \alpha''_i$ we have $\bar{x}' = \varepsilon$, $\alpha'_i \in \{true, false\}$ and $\alpha''_i \in \{true, false\}$,
- (**) for every formula $\exists \bar{y}'_{ij} \beta'_{ij}$ we have $\bar{y}'_{ij} = \varepsilon$ and $\beta'_{ij} \in \{true, false\}$.

According to the condition 4 of (3.21), all the α'_i and α''_i are different from *false*, thus according to (*) we get

- (***) The formulas $\exists \bar{x}'_i \alpha'_i \wedge \alpha''_i$ are of the form $\exists \varepsilon\, true \wedge true$.

On the other hand, according to (**) and the condition 5 of (3.21), we deduce that the sets J_i of (3.21) are empty. Thus, according to (***) we deduce that φ is of the form

$$true \wedge (\bigwedge_{i\in I} \neg(\exists \varepsilon\, true \wedge true))$$

If $I = \emptyset$ then φ is the formula *true*, else, since we do not distinguish two formulas which can be made equal using the following transformations of sub-formulas

$$\varphi \wedge \psi \Longrightarrow \psi \wedge \varphi, \quad (\varphi \wedge \psi) \wedge \phi \Longrightarrow \varphi \wedge (\psi \wedge \phi),$$
$$\varphi \wedge true \Longrightarrow \varphi, \quad \varphi \vee false \Longrightarrow \varphi,$$

then φ is the formula

$$\bigwedge \neg true.$$

□

Property 3.3.2.5 *Every solved formula is equivalent in T to wnfv boolean combination of formulas of the form $\exists \bar{x}'\, \alpha' \wedge \alpha''$ with $\exists \bar{x}'\, \alpha' \in A'$ and $\alpha'' \in A''$.*

Proof. Let φ be a solved formula. According to Definition 3.3.2.3, the formula φ is of the form

$$\neg(\exists \bar{x}'\, \alpha' \wedge \alpha'' \bigwedge_{i \in I} \neg(\exists \bar{y}_i'\, \beta_i')),$$

with $\exists \bar{x}'\alpha' \in A'$, $\alpha'' \in A''$ and $\exists \bar{y}_i'\beta_i' \in A'$ for all $i \in I$. Since $\exists \bar{x}'\alpha' \in A'$ then according to Definition 3.2.1.1, $T \models \exists ?\bar{x}'\alpha'$ and thus $T \models \exists ?\bar{x}'\alpha' \wedge \alpha''$. Then, according to Corollary 3.1.0.7, φ is equivalent in T to

$$\neg((\exists \bar{x}'\, \alpha' \wedge \alpha'') \wedge \bigwedge_{i \in I} \neg(\exists \bar{x}'\, \alpha' \wedge \alpha'' \wedge (\exists \bar{y}_i'\, \beta_i'))).$$

According to the definition of working formula, all the quantified variables of φ have distinct names and different from those of the free variables, thus the preceding formula is equivalent in T to

$$\neg((\exists \bar{x}'\, \alpha' \wedge \alpha'') \wedge \bigwedge_{i \in I} \neg(\exists \bar{x}'\bar{y}_i'\, \alpha' \wedge \alpha'' \wedge \beta_i')).$$

Since $\exists \bar{x}'\alpha' \in A'$, $\alpha'' \in A''$ and $\exists \bar{y}_i'\beta_i' \in A'$ for all $i \in I$, then $\alpha' \in A$, $\alpha'' \in A$ and $\beta_i' \in A$. Since A is closed for the conjunction, then $\alpha' \wedge \alpha'' \wedge \beta_i' \in A$ for all $i \in I$. According to Property 3.2.2.1, the preceding formula is equivalent in T to a wnfv formula of the form

$$\neg((\exists \bar{x}'\, \alpha' \wedge \alpha'') \wedge \bigwedge_{i \in I} \neg(\exists \bar{y}_i'\, \beta_i' \wedge \beta_i'')),$$

with $\exists \bar{x}'\alpha' \in A'$, $\alpha'' \in A''$ and for all $i \in I$ we have $\exists \bar{y}_i'\, \beta_i' \in A'$ and $\beta_i'' \in A''$. The preceding formula is finally equivalent in T to

$$(\neg(\exists \bar{x}'\, \alpha' \wedge \alpha'')) \vee \bigvee_{i \in I} (\exists \bar{y}_i'\, \beta_i' \wedge \beta_i'').$$

which is a boolean combination wnfv of elements of the form $\exists \bar{x}'\, \alpha' \wedge \alpha''$ with $\exists \bar{x}'\, \alpha' \in A'$ and $\alpha'' \in A''$. □

3.3.3 The rewriting rules

We present now the rewriting rules which transform a conjunction φ of working formulas of any depth d into a wnfv conjunction ϕ of solved formulas which is equivalent to φ in T. To apply the rule $p_1 \Longrightarrow p_2$ to the working formula p means to replace in p, a sub-formula p_1 by the formula

p_2, by considering that the connector $\wedge$ is associative and commutative.

$$(1)\quad \neg\begin{bmatrix}\exists\bar{x}\,\alpha\wedge\varphi\wedge\\ \neg(\exists\bar{y}\,true)\end{bmatrix} \implies true$$

$$(2)\quad \neg\begin{bmatrix}\exists\bar{x}\,\alpha\wedge false\wedge\varphi\end{bmatrix} \implies true$$

$$(3)\quad \neg\begin{bmatrix}\exists\bar{x}\,\alpha\wedge\\ \bigwedge_{i\in I}\neg(\exists\bar{y}_i\,\beta_i)\end{bmatrix} \implies \neg\begin{bmatrix}\exists\bar{x}'\bar{x}''\,\alpha'\wedge\alpha''\wedge\\ \bigwedge_{i\in I}\neg(\exists\bar{x}'''\bar{y}_i\,\alpha'''\wedge\beta_i)^*\end{bmatrix}$$

$$(4)\quad \neg\begin{bmatrix}\exists\bar{x}\,\alpha\wedge\varphi\wedge\\ \neg(\exists\bar{y}'\,\beta'\wedge\beta'')\end{bmatrix} \implies \begin{bmatrix}\neg(\exists\bar{x}\,\alpha\wedge\varphi\wedge\neg(\exists\bar{y}'\,\beta'))\wedge\\ \bigwedge_{i\in I}\neg(\exists\bar{x}\bar{y}'\,\alpha\wedge\beta'\wedge\beta''_i\wedge\varphi)^*\end{bmatrix}$$

$$(5)\quad \neg\begin{bmatrix}\exists\bar{x}\,\alpha\wedge\\ \bigwedge_{i\in I}\neg(\exists\bar{y}'_i\,\beta'_i)\end{bmatrix} \implies \neg\begin{bmatrix}\exists\bar{x}'\,\alpha'\wedge\alpha''_*\\ \bigwedge_{i\in I'}\neg(\exists\bar{y}'_i\,\beta'_i)\end{bmatrix}$$

$$(6)\quad \neg\begin{bmatrix}\exists\bar{x}\,\alpha\wedge\varphi\wedge\\ \neg\begin{bmatrix}\exists\bar{y}'\,\beta'\wedge\beta''\wedge\\ \bigwedge_{i\in I}\neg(\exists\bar{z}'_i\,\delta'_i)\end{bmatrix}\end{bmatrix} \implies \begin{bmatrix}\neg(\exists\bar{x}\,\alpha\wedge\varphi\wedge\neg(\exists\bar{y}'\,\beta'\wedge\beta''))\wedge\\ \bigwedge_{i\in I}\neg(\exists\bar{x}\bar{y}'\bar{z}_i\,\alpha\wedge\beta'\wedge\beta''\wedge\delta'_i\wedge\varphi)^*\end{bmatrix}$$

with α an element of A, φ a conjunction of working formulas and I a finite possibly empty set. In the rule (3), the formula $\exists\bar{x}\,\alpha$ is equivalent in T to a decomposed formula of the form $\exists\bar{x}'\,\alpha'\wedge(\exists\bar{x}''\,\alpha''\wedge(\exists\bar{x}'''\,\alpha'''))$ with $\exists\bar{x}'\,\alpha'\in A'$, $\alpha''\in A''$, $\alpha'''\in A$, $T\models\forall\bar{x}''\alpha''\rightarrow\exists!\bar{x}'''\alpha'''$ and $\exists\bar{x}'''\,\alpha'''$ is different from $\exists\varepsilon\,true$. All the β_i belong to A. The formula $(\exists\bar{x}'''\bar{y}_i\,\alpha'''\wedge\beta_i)^*$ is the formula $(\exists\bar{x}'''\bar{y}_i\,\alpha'''\wedge\beta_i)$ in which we have renamed the variables which occur in $\bar{x}'''$ by distinct names and different from those of the free variables. In the rule (4), the formula $\exists\bar{x}\,\alpha$ is equivalent in T to a decomposed formula of the form $\exists\bar{x}'\,\alpha'\wedge(\exists\bar{x}''\,\alpha''\wedge(\exists\varepsilon\,true))$ with $\exists\bar{x}'\,\alpha'\in A'$ and $\alpha''\in A''$. The formula $\exists\bar{y}'\,\beta'$ belongs to A'. The formula β'' belongs to A'' and is different from the formula *true*. Moreover, $T\models(\neg\beta'')\leftrightarrow\bigvee_{i\in I}\beta''_i$ with $\beta''_i\in A$. The formula $(\exists\bar{x}\bar{y}'\,\alpha\wedge\beta'\wedge\beta''_i\wedge\varphi)^*$ is the formula $(\exists\bar{x}\bar{y}'\,\alpha\wedge\beta'\wedge\beta''_i\wedge\varphi)$ in which we have renamed the variables which occur in $\bar{x}$ and $\bar{y}'$ by distinct names and different from those of the free variables. In the rule (5), the formula $\exists\bar{x}\,\alpha$ is not of the form $\exists\bar{x}\,\alpha_1\wedge\alpha_2$ with $\exists\bar{x}\,\alpha_1\in A'$ and $\alpha_2\in A''$, and is equivalent in T to a decomposed formula of the form $\exists\bar{x}'\,\alpha'\wedge(\exists\bar{x}''\,\alpha''\wedge(\exists\varepsilon\,true))$ with $\exists\bar{x}'\,\alpha'\in A'$ and $\alpha''\in A''$. Each formula $\exists\bar{y}'_i\,\beta'_i$ belongs to A'. The set I' is the set of the $i\in I$ such that $\exists\bar{y}'_i\beta'_i$ has no occurrences of any variable of $\bar{x}''$. Moreover, $T\models(\exists\bar{x}''\alpha'')\leftrightarrow\alpha''_*$ with $\alpha''_*\in A''$. In the rule (6), $I\neq\emptyset$, $\exists\bar{y}'\,\beta'\in A'$, $\exists\bar{z}'_i\,\delta'_i\in A'$ and $\beta''\in A''$. The formula $(\exists\bar{x}\bar{y}'\bar{z}_i\,\alpha\wedge\beta'\wedge\beta''\wedge\delta'_i\wedge\varphi)^*$ is the formula $(\exists\bar{x}\bar{y}'\bar{z}_i\,\alpha\wedge\beta'\wedge\beta''\wedge\delta'_i\wedge\varphi)$ in which we have renamed the variables which occur in $\bar{x}$ and $\bar{y}'$ by distinct names and different from those of the free variables.

Property 3.3.3.1 *Every repeated application of our rewriting rules on a conjunction φ of working formulas terminates and produces a wnfv conjunction ϕ of solved formulas equivalent to φ in T.*

Proof, first part : Let us show that for each rule of the form $p\implies p'$ we have $T\models p\leftrightarrow p'$ and the formula p' remains a conjunction of working formulas. It is clear that the rules 1 and 2 are correct in T.

Correctness of the rule (3):

$$\neg\begin{bmatrix}\exists\bar{x}\,\alpha\wedge\\ \bigwedge_{i\in I}\neg(\exists\bar{y}_i\,\beta_i)\end{bmatrix}\Longrightarrow\neg\begin{bmatrix}\exists\bar{x}'\bar{x}''\,\alpha'\wedge\alpha''\wedge\\ \bigwedge_{i\in I}\neg(\exists\bar{x}'''\bar{y}_i\,\alpha'''\wedge\beta_i)^*\end{bmatrix}$$

where the formula $\exists\bar{x}\,\alpha$ is equivalent in T to a decomposed formula of the form $\exists\bar{x}'\,\alpha'\wedge(\exists\bar{x}''\,\alpha''\wedge(\exists\bar{x}'''\,\alpha'''))$ with $\exists\bar{x}'\,\alpha'\in A'$, $\alpha''\in A''$, $\alpha'''\in A$, $T\models\forall\bar{x}''\alpha''\rightarrow\exists!\bar{x}'''\alpha'''$ and $\exists\bar{x}'''\,\alpha'''$ different from $\exists\varepsilon\,true$. The formula $(\exists\bar{x}'''\bar{y}_i\,\alpha'''\wedge\beta_i)^*$ is the formula $(\exists\bar{x}'''\bar{y}_i\,\alpha'''\wedge\beta_i)$ in which we have renamed the variables which occur in $\bar{x}'''$ by distinct names and different from those of the free variables.

Let us show the correctness of this rule. According to the conditions of this rule, the formula $\exists\bar{x}\,\alpha$ is equivalent in T to a decomposed formula of the form $\exists\bar{x}'\,\alpha'\wedge(\exists\bar{x}''\,\alpha''\wedge(\exists\bar{x}'''\,\alpha'''))$ with $\exists\bar{x}'\,\alpha'\in A'$, $\alpha''\in A''$, $\alpha'''\in A$, $T\models\forall\bar{x}''\alpha''\rightarrow\exists!\bar{x}'''\alpha'''$ and $\exists\bar{x}'''\,\alpha'''$ different from $\exists\varepsilon\,true$. Thus the left hand side of this rule is equivalent in T to

$$\neg(\exists\bar{x}'\,\alpha'\wedge(\exists\bar{x}''\alpha''\wedge(\exists\bar{x}'''\alpha'''\wedge\bigwedge_{i\in I}\neg(\exists\bar{y}_i\,\beta_i)))).$$

According to Corollary 3.1.0.8, the preceding formula is equivalent in T to

$$\neg(\exists\bar{x}'\,\alpha'\wedge(\exists\bar{x}''\alpha''\wedge\bigwedge_{i\in I}\neg(\exists\bar{x}'''\alpha'''\wedge(\exists\bar{y}_i\,\beta_i)))).$$

According to the definition of working formula, the quantified variables have distinct names and different from those of the free variables. We can then lift the quantifications $\exists\bar{y}_i$. The preceding formula is thus equivalent in T to

$$\neg(\exists\bar{x}'\,\alpha'\wedge(\exists\bar{x}''\alpha''\wedge\bigwedge_{i\in I}\neg(\exists\bar{x}'''\bar{y}_i\,\alpha'''\wedge\beta_i))),$$

which, by renaming the variables which occur in $\bar{x}'''$ by distinct names and different from those of the free variables, is equivalent in T to

$$\neg(\exists\bar{x}'\,\alpha'\wedge(\exists\bar{x}''\alpha''\wedge\bigwedge_{i\in I}\neg(\exists\bar{x}'''\bar{y}_i\,\alpha'''\wedge\beta_i)^*)),$$

thus, the rule (3) is correct in T.

Correctness of the rule (4):

$$\neg\begin{bmatrix}\exists\bar{x}\,\alpha\wedge\varphi\wedge\\ \neg(\exists\bar{y}'\,\beta'\wedge\beta'')\end{bmatrix}\Longrightarrow\begin{bmatrix}\neg(\exists\bar{x}\,\alpha\wedge\varphi\wedge\neg(\exists\bar{y}'\,\beta'))\wedge\\ \bigwedge_{i\in I}\neg(\exists\bar{x}\bar{y}'\,\alpha\wedge\beta'\wedge\beta''_i\wedge\varphi)^*\end{bmatrix}$$

where the formula $\exists\bar{x}\,\alpha$ is equivalent in T to a decomposed formula of the form $\exists\bar{x}'\,\alpha'\wedge(\exists\bar{x}''\,\alpha''\wedge(\exists\varepsilon\,true))$ with $\exists\bar{x}'\,\alpha'\in A'$ and $\alpha''\in A''$. The formula $\exists\bar{y}'\,\beta'$ belongs to A'. The formula β'' belongs to A'' and is not of the form $true$. Moreover, $T\models(\neg\beta'')\leftrightarrow\bigvee_{i\in I}\beta''_i$ with $\beta''_i\in A$. The formula $(\exists\bar{x}\bar{y}'\,\alpha\wedge\beta'\wedge\beta''_i\wedge\varphi)^*$ is the formula $(\exists\bar{x}\bar{y}'\,\alpha\wedge\beta'\wedge\beta''_i\wedge\varphi)$ in which we have renamed the variables which occur in $\bar{x}$ and $\bar{y}'$ by distinct names and different from those of the free variables.

Since $\exists\bar{y}'\beta'\in A'$, then according to the second point of Definition 3.2.1.1, we have $T\models\exists?\bar{y}'\beta'$, thus according to Corollary 3.1.0.6, the left hand side of our rule is equivalent in T to

$$\neg\begin{bmatrix}(\exists\bar{x}\,\alpha\wedge\varphi\wedge\neg(\exists\bar{y}'\,\beta'))\vee\\ (\exists\bar{x}\bar{y}'\,\alpha\wedge\varphi\wedge\beta'\wedge\neg\beta'')\end{bmatrix}.$$

Since $T \models (\neg\beta'') \leftrightarrow (\bigvee_{i\in I} \beta''_i)$ (always possible according to the condition 3 of Definition 3.2.1.1), then the preceding formula is equivalent in T to

$$\neg \begin{bmatrix} (\exists\bar{x}\, \alpha \wedge \varphi \wedge \neg(\exists\bar{y}'\, \beta'))\vee \\ (\exists\bar{x}\bar{y}'\, \alpha \wedge \varphi \wedge \beta' \wedge (\bigvee_{i\in I} \beta''_i)) \end{bmatrix},$$

i.e. to

$$\neg \begin{bmatrix} (\exists\bar{x}\, \alpha \wedge \varphi \wedge \neg(\exists\bar{y}'\, \beta'))\vee \\ (\exists\bar{x}\bar{y}'\, \bigvee_{i\in I}(\alpha \wedge \varphi \wedge \beta' \wedge \beta''_i)) \end{bmatrix},$$

i.e. to

$$\neg \begin{bmatrix} (\exists\bar{x}\, \alpha \wedge \varphi \wedge \neg(\exists\bar{y}'\, \beta'))\vee \\ \bigvee_{i\in I}(\exists\bar{x}\bar{y}'\, \alpha \wedge \beta' \wedge \beta''_i \wedge \varphi) \end{bmatrix},$$

and thus to

$$\begin{bmatrix} \neg(\exists\bar{x}\, \alpha \wedge \varphi \wedge \neg(\exists\bar{y}'\, \beta'))\wedge \\ \bigwedge_{i\in I} \neg(\exists\bar{x}\bar{y}'\, \alpha \wedge \beta' \wedge \beta''_i \wedge \varphi) \end{bmatrix},$$

which, by denoting by $(\exists\bar{x}\bar{y}'\, \alpha \wedge \beta' \wedge \beta''_i \wedge \varphi)^*$ the formula $(\exists\bar{x}\bar{y}'\, \alpha \wedge \beta' \wedge \beta''_i \wedge \varphi)$ in which we have renamed the variables which occur in $\bar{x}$ and $\bar{y}'$ by distinct names and different from those of the free variables, is equivalent in T to

$$\begin{bmatrix} \neg(\exists\bar{x}\, \alpha \wedge \varphi \wedge \neg(\exists\bar{y}'\, \beta'))\wedge \\ \bigwedge_{i\in I} \neg(\exists\bar{x}\bar{y}'\, \alpha \wedge \beta' \wedge \beta''_i \wedge \varphi)^* \end{bmatrix}.$$

Thus, the rule (4) is correct in T.

Correctness of the rule (5):

$$\neg \begin{bmatrix} \exists\bar{x}\, \alpha\wedge \\ \bigwedge_{i\in I} \neg(\exists\bar{y}'_i\, \beta'_i) \end{bmatrix} \Longrightarrow \neg \begin{bmatrix} \exists\bar{x}'\, \alpha' \wedge \alpha''_* \\ \bigwedge_{i\in I'} \neg(\exists\bar{y}'_i\, \beta'_i) \end{bmatrix}$$

where the formula $\exists\bar{x}\, \alpha$ is not of the form $\exists\bar{x}\, \alpha_1 \wedge \alpha_2$ with $\exists\bar{x}\, \alpha_1 \in A'$ and $\alpha_2 \in A''$ and is equivalent in T to a decomposed formula of the form $\exists\bar{x}'\, \alpha' \wedge (\exists\bar{x}''\, \alpha'' \wedge (\exists\varepsilon\, true))$ with $\exists\bar{x}'\, \alpha' \in A'$, $\alpha'' \in A''$. Each formula $\exists\bar{y}'_i\, \beta'_i$ belongs to A'. I' is the set of the $i \in I$ such that $\exists\bar{y}'_i\beta'_i$ has no occurrences of the variables of $\bar{x}''$. Moreover, $T \models (\exists\bar{x}''\alpha'') \leftrightarrow \alpha''_*$ with $\alpha''_* \in A''$.

Let us show the correctness of this rule. According to the conditions of this rule, its left hand side is equivalent in T to

$$\neg(\exists\bar{x}'\, \alpha' \wedge (\exists\bar{x}''\alpha'' \wedge \bigwedge_{i\in I} \neg(\exists\bar{y}'_i\, \beta'_i))),$$

with $\exists\bar{x}'\, \alpha' \in A'$, $\alpha'' \in A''$ and all the $\exists\bar{y}'_i\, \beta'_i$ belong to A'. Let us denote by I_1, the set of the $i \in I$ such that x''_n has no occurrences in $\exists\bar{y}'_i\beta'_i$. The preceding formula is equivalent in T to

$$\neg(\exists\bar{x}'\alpha' \wedge (\exists x''_1 ... \exists x''_{n-1} \begin{bmatrix} (\bigwedge_{i\in I_1} \neg(\exists\bar{y}'_i\beta'_i))\wedge \\ (\exists x''_n\, \alpha'' \wedge \bigwedge_{i\in I-I_1} \neg(\exists\bar{y}'_i\beta'_i)) \end{bmatrix})). \quad (3.22)$$

Since $\alpha'' \in A''$ and $\exists\bar{y}'_i\beta'_i \in A'$, then according to Property 3.1.0.4 and the conditions 2 and 3 of Definition 3.2.1.1, the formula (3.22) is equivalent in T to

$$\neg(\exists\bar{x}'\alpha' \wedge (\exists x''_1 ... \exists x''_{n-1} \begin{bmatrix} (\bigwedge_{i\in I_1} \neg(\exists\bar{y}'_i\beta'_i))\wedge \\ (\exists x''_n\, \alpha'') \end{bmatrix})).$$

Since $T \models (\exists x''_n \alpha'') \leftrightarrow \alpha''_n$ with $\alpha''_n \in A''$ (always possible according to the condition 3 of Definition 3.2.1.1), then the preceding formula is equivalent in T to

$$\neg(\exists \bar{x}' \alpha' \wedge (\exists x''_1 ... \exists x''_{n-1}((\bigwedge_{i \in I_1} \neg(\exists \bar{y}'_i \beta'_i)) \wedge \alpha''_n))), \tag{3.23}$$

thus to

$$\neg(\exists \bar{x}' \alpha' \wedge (\exists x''_1 ... \exists x''_{n-1}\, \alpha''_n \wedge \bigwedge_{i \in I_1} \neg(\exists \bar{y}'_i \beta'_i))). \tag{3.24}$$

By repeating the four last steps $(n-1)$ times and by denoting by I_k the set of the $i \in I_{k-1}$ such that $x''_{(n-k+1)}$ has no occurrences in $\exists \bar{y}'_i \beta'_i$, the preceding formula is equivalent in T to

$$\neg(\exists \bar{x}' \alpha' \wedge \alpha''_1 \wedge \textstyle\bigwedge_{i \in I_n} \neg(\exists \bar{y}'_i \beta'_i)).$$

Thus, the rule (5) is correct in T.

Correctness of the rule (6):

$$\neg \begin{bmatrix} \exists \bar{x}\, \alpha \wedge \varphi \wedge \\ \neg \begin{bmatrix} \exists \bar{y}'\, \beta' \wedge \beta'' \\ \bigwedge_{i \in I} \neg(\exists \bar{z}'_i\, \delta'_i) \end{bmatrix} \end{bmatrix} \Longrightarrow \begin{bmatrix} \neg(\exists \bar{x}\, \alpha \wedge \varphi \wedge \neg(\exists \bar{y}'\, \beta' \wedge \beta'')) \wedge \\ \bigwedge_{i \in I} \neg(\exists \bar{x} \bar{y}' \bar{z}'_i\, \alpha \wedge \beta' \wedge \beta'' \wedge \delta'_i \wedge \varphi)^* \end{bmatrix}$$

where $I \neq \emptyset$, $\exists \bar{y}'\, \beta' \in A'$, $\beta'' \in A''$ and $\exists \bar{z}'_i\, \delta'_i \in A'$. The formula $(\exists \bar{x} \bar{y}' \bar{z}_i\, \alpha \wedge \beta' \wedge \beta'' \wedge \delta'_i \wedge \varphi)^*$ is the formula $(\exists \bar{x} \bar{y}' \bar{z}_i\, \alpha \wedge \beta' \wedge \beta'' \wedge \delta'_i \wedge \varphi)$ in which we have renamed the variables which occur in $\bar{x}$ and $\bar{y}'$ by distinct names and different from those of the free variables.

Let us show the correctness of this rule. Since $\exists \bar{y}' \beta' \in A'$, then according to the second point of Definition 3.2.1.1, we have $T \models \exists ? \bar{y}' \beta'$, thus $T \models \exists ? \bar{y}' \beta' \wedge \beta''$. Thus, according to Corollary 3.1.0.7, the left hand side of this rule is equivalent in T to

$$\neg \begin{bmatrix} \exists \bar{x}\, \alpha \wedge \varphi \wedge \\ \neg \left[\ (\exists \bar{y}'\, \beta' \wedge \beta'') \wedge \bigwedge_{i \in I} \neg(\exists \bar{y}'\, \beta' \wedge \beta'' \wedge (\exists \bar{z}'_i\, \delta'_i)) \ \right] \end{bmatrix},$$

i.e. to

$$\neg \begin{bmatrix} \exists \bar{x}\, \alpha \wedge \varphi \wedge \\ \neg \left[\ (\exists \bar{y}'\, \beta' \wedge \beta'') \wedge \bigwedge_{i \in I} \neg(\exists \bar{y}' \bar{z}'_i\, \beta' \wedge \beta'' \wedge \delta'_i) \ \right] \end{bmatrix},$$

thus to

$$\neg \begin{bmatrix} \exists \bar{x}'\, \alpha \wedge \varphi \wedge \\ \left[\ (\neg(\exists \bar{y}'\, \beta' \wedge \beta'')) \vee \bigvee_{i \in I} (\exists \bar{y}' \bar{z}'_i\, \beta' \wedge \beta'' \wedge \delta'_i) \ \right] \end{bmatrix}.$$

After having distributed the $\wedge$ on the $\vee$ and lifted the quantifications $\exists \bar{y}' \bar{z}'_i$ we get

$$\neg \begin{bmatrix} (\exists \bar{x}\, \alpha \wedge \varphi \wedge \neg(\exists \bar{y}'\, \beta' \wedge \beta'')) \vee \\ \bigvee_{i \in I} (\exists \bar{x} \bar{y}' \bar{z}'_i\, \alpha \wedge \varphi \wedge \beta' \wedge \beta'' \wedge \delta'_i) \end{bmatrix},$$

which is equivalent in T to

$$\begin{bmatrix} \neg(\exists \bar{x}\, \alpha \wedge \varphi \wedge \neg(\exists \bar{y}'\, \beta' \wedge \beta'')) \wedge \\ \bigwedge_{i \in I} \neg(\exists \bar{x} \bar{y}' \bar{z}'_i\, \alpha \wedge \varphi \wedge \beta' \wedge \beta'' \wedge \delta'_i) \end{bmatrix},$$

which, by denoting by $(\exists \bar{x}\bar{y}'\bar{z}'_i\, \alpha \wedge \varphi \wedge \beta' \wedge \beta'' \wedge \delta'_i)$ the formula $(\exists \bar{x}\bar{y}'\bar{z}'_i\, \alpha \wedge \varphi \wedge \beta' \wedge \beta'' \wedge \delta'_i)$ in wich we have renamed the variables which occur in $\bar{x}$ and $\bar{y}'$ by distinct names and different from those of the free variables, is equivalent in T to

$$\left[\begin{array}{l} \neg(\exists \bar{x}\, \alpha \wedge \varphi \wedge \neg(\exists \bar{y}'\, \beta' \wedge \beta''))\wedge \\ \bigwedge_{i\in I} \neg(\exists \bar{x}\bar{y}'\bar{z}'_i\, \alpha \wedge \varphi \wedge \beta' \wedge \beta'' \wedge \delta'_i)^* \end{array}\right].$$

Thus, the rule (6) is correct in T.

Proof, second part: every finite application of the rules on a conjunction of working formulas produces a conjunction of solved formulas.

Let us show first that every substitution of a sub-working formula of a conjunction of working formulas by a conjunction of working formulas produces a conjunction of working formulas. Let $\bigwedge_{i\in I}\varphi_i$ be a conjunction of working formulas. Let φ_k, with $k \in I$, be an element of this conjunction of depth d_k. Two cases arise:

1. Either we replace φ_k by a conjunction of working formulas. Thus, let $\bigwedge_{j\in J_k}\phi_j$ be a conjunction of working formulas which is equivalent to φ_k in T. The conjunction of working formulas $\bigwedge_{i\in I}\varphi_i$ is equivalent in T to

$$(\bigwedge_{i\in I-\{k\}} \varphi_i) \wedge (\bigwedge_{j\in J_k} \phi_j)$$

   which is clearly a conjunction of working formulas.

2. Or, we replace a strict sub-working formula of φ_k by a conjunction of working formulas. Thus, let ϕ be a sub-working formula of φ_k of depth $d_\phi < d_k$ (thus ϕ is different from φ_k). Thus, φ_k has a sub-working formula[16] of the form

$$\neg(\exists \bar{x}\alpha \wedge (\bigwedge_{l\in L} \psi_l) \wedge \phi),$$

   where L is a finite (possibly empty) set and all the ψ_l are working formulas. Let $\bigwedge_{j\in J}\phi_j$ be a conjunction of working formulas which is equivalent to ϕ in T. Thus, the preceding sub-working formula of φ_k is equivalent in T to

$$\neg(\exists \bar{x}\alpha \wedge (\bigwedge_{l\in L} \psi_l) \wedge (\bigwedge_{j\in J} \phi_j)),$$

   which is clearly a sub-working formula. Thus, φ_k is equivalent to a working formula and thus $\bigwedge_{i\in I}\varphi_i$ is equivalent to a conjunction of working formulas.

From 1 and 2 we deduce that (i) every substitution of a sub-working formula of a conjunction of working formulas by a conjunction of working formulas produces a conjunction of working formulas.

Since each rule transforms a working formula into a conjunction of working formulas, then according to (i) every finite application of the rewriting rules on a conjunction of working formulas produces a conjunction of working formulas. Let us show now that each of these final working formulas is solved.

Let φ be a working formula. We have shown that every finite application of the rules on a conjunction of working formulas produces a conjunction ϕ of working formulas. Suppose that the

[16] By considering that the set of the sub-formulas of any formula φ contains also the whole formula φ.

rules terminate and one at least of the working formulas of ϕ is not solved. Let ψ this working formula. Two cases arise

Case 1: ψ is a working formula of depth greater than 2. Thus ψ contains a sub-formula of the form

$$\neg\left[\begin{array}{l}\exists\bar{x}\,\alpha\wedge\psi_1\wedge\\ \neg\left[\ \exists\bar{y}\,\beta\wedge\bigwedge_{i\in I}\neg(\exists\bar{z}_i\,\delta_i)\ \right]\end{array}\right],$$

where ψ_1 is a conjunction of working formulas, I is a finite non-empty set and α, β, and δ_i belong to A. Let $(\exists\bar{y}'\beta'\wedge(\exists\bar{x}''\beta''\wedge(\exists\bar{y}'''\beta''')))$ be the decomposed formula of $\exists\bar{y}\beta$ and let $(\exists\bar{z}_i'\delta_i'\wedge(\exists\bar{z}_i''\delta_i''\wedge(\exists\bar{z}_i'''\delta_i''')))$ be the decomposed formula of $\exists\bar{z}_i\delta_i$. If $\exists\bar{y}'''\beta'''$ is different from the formula $\exists\varepsilon true$ then the rule (3) can be applied anymore which contradicts our suppositions. Thus, suppose that

$$\exists\bar{y}'''\beta'''=\exists\varepsilon true. \tag{3.25}$$

If there exists $k\in I$ such that $\exists\bar{z}_k'''\delta_k'''$ is not the formula $\exists\varepsilon true$, then the rule (3) can still be applied which contradicts our suppositions. Thus, suppose that

$$\exists\bar{z}_i'''\delta_i'''=\exists\varepsilon true, \tag{3.26}$$

for all $i\in I$. If there exists $k\in I$ such that $\exists\bar{z}_k\beta_k$ is not of the form $\exists\bar{z}_k'\beta_k'\wedge\beta_k''$ with $\exists\bar{z}_k'\beta_k'\in A'$ and $\beta_k''\in A''$, then since we have (3.26), the rule (5) can still be applied which contradicts our supposition. Suppose that

$$each\ \exists\bar{z}_i\delta_i\ is\, of\, the\, form\ \exists\bar{z}_i'\delta_i'\wedge\delta_i''\ with\,\exists\bar{z}_i'\delta_i'\in A'\ and\,\delta_i''\in A''. \tag{3.27}$$

If there exists $i\in I$ such that δ_i'' is not the formula *true*, then since we have (3.25), the rule (4) can still be applied which contradicts our suppositions. Suppose that all the δ_i''' are of the form *true*. According to (3.27) we get

$$all\ the\ \exists\bar{z}_i\beta_i\ belong\ to\ A'. \tag{3.28}$$

If $\exists\bar{y}\beta$ is not of the form $\exists\bar{y}'\beta'\wedge\beta''$ with $\exists\bar{y}'\beta'\in A'$ and $\beta''\in A''$, then since we have (3.25) and (3.28), the rule (5) can still be applied which contradicts our suppositions. Thus suppose that

$$\exists\bar{y}\beta\ is\ of\ the\ form\ \exists\bar{y}'\beta'\wedge\beta''\ with\ \exists\bar{y}'\beta'\in A'\ et\ \beta''\in A''. \tag{3.29}$$

Since we have (3.29) and (3.28), the rule (6) can still be applied which contradicts all our suppositions.

Case 2: ψ is a working formula of the form

$$\exists\bar{x}\,\alpha\wedge\bigwedge_{i\in I}\neg(\exists\bar{y}_i\,\beta_i)$$

and at least one of the following properties holds

1. *false* is a sub-formula of α,
2. there exists $i\in I$ such that β_i is the formula *true* or *false*,
3. there exists $i\in I$ such that $\exists\bar{y}_i\beta_i\notin A'$,
4. $\exists\bar{x}\alpha$ is not of the form $\exists\bar{x}'\alpha'\wedge\alpha''$ with $\exists\bar{x}'\alpha'\in A'$ and $\alpha''\in A''$.

if the condition (1) holds then the rule (2) can still be applied which contradicts our suppositions. If the condition (2) holds then the rules (1) and (2) can still be applied which contradicts our suppositions. If the condition (3) holds then the rules (3) or (4) or (5) (with $I = \emptyset$) can still be applied which contradicts our suppositions. If the condition (4) holds then according to the preceding point $\exists \bar{y}_i \beta_i \in A'$ and thus the rule (3) or (5) can still be applied which contradicts our suppositions.

From the cases 1 and 2, our suppositions are always false, thus ψ is a solved formula and thus ϕ is a conjunction of solved formulas.

Proof, third part: Let us show that every repeated application of the preceding rules terminates. The termination of the rule is intuitive and can be shown using the same functions used in the rules of infinite-decomposable theories. Nevertheless, the new rule (4) needs a function much more complex than the function β defined in Chapter 3. We prefer then giving a semi-formal proof using the conditions of applications of our rules. Let φ be a working formula. The rule (3) treats first the two most embedded levels of φ using decompositions. Then, the rule (5) eliminates the quantifications of the $\bar{x}''$ on the last levels. Then the rule (4) removes the formulas of A''. After many applications of the preceding steps, the rule (6) decreases the depth of φ. All these steps are repeated until reaching the conjunction of solved formulas. □

3.3.4 The decision procedure

Having a proposition ψ, the decision of ψ in T proceeds as follows:

1. Transform the formula ψ into a normalized formula, then into a conjunction of working formulas φ wnfv and equivalent to ψ in T.

2. Apply the rules on φ as many times as possible. At the end, we get a conjunction ϕ of solved formulas.

Since the transformation of the proposition ψ into a conjunction of working formulas φ is wnfv, then φ is a conjunction of working formulas without free variables. According to Property 3.3.3.1, the application of our rules on φ produces a wnfv conjunction ϕ of solved formulas and thus a conjunction ϕ of solved formulas without free variables. According to Property 3.3.2.4, ϕ is either the formula *true*, or the formula $\bigwedge_{i \in I} \neg true$. Since T is decomposable, it has at least one model and thus either $T \models \phi$, or $T \models \neg\phi$ and thus either $T \models \psi$, or $T \models \neg\psi$. This algorithm can be applied on formulas having free variables and produces in this case a conjunction of solved formulas having free variables. According to Property 3.3.2.5 we have:

Corollary 3.3.4.1 *If T is zero-infinite-decomposable then every formula is equivalent in T either to true, or to false, or to a boolean combination of formulas of the form $\exists \bar{x}'\, \alpha' \wedge \alpha''$ with $\exists \bar{x}'\, \alpha' \in A'$, $\alpha'' \in A''$ and having at least one free variable.*

This corollary is another proof of the completeness of the zero-infinite-decomposable theories.

3.4 Application to the construction of trees on an ordered set $\mathcal{T}_{ord}$

3.4.1 Axiomatization

Let F be an **infinite** set of function symbols each one having a non-nul arity. Let R be a set of relation symbols containing the relation symbols $<$ and *num* of respective arities $2, 1$. If t_1 and t_2 are terms, then we write $t_1 < t_2$ for $< (t_1, t_2)$. The construction of trees on an ordered set, denoted by $\mathcal{T}_{ord}$, is the first order theory built on the signature $F \cup R$ and whose axioms are the following propositions:

1 $\forall \bar{x} \forall \bar{y}\, f\bar{x} = f\bar{y} \rightarrow \bigwedge_i x_i = y_i$,
2 $\forall \bar{x} \forall \bar{y}\, \neg f\bar{x} = g\bar{y}$,
3 $\forall \bar{x} \exists! \bar{z}\, \bigwedge_i z_i = t_i(\bar{z}, \bar{x})$,
4 $\forall x\, num\, x \rightarrow \neg x < x$,
5 $\forall x \forall y \forall z\, num\, x \wedge num\, y \wedge num\, z \rightarrow ((x < y \wedge y < z) \rightarrow x < z)$,
6 $\forall x \forall y\, (num\, x \wedge num\, y) \rightarrow (x < y \vee x = y \vee y < x)$,
7 $\forall x \forall y\, (num\, x \wedge num\, y) \rightarrow (x < y \rightarrow (\exists z\, num\, z \wedge x < z \wedge z < y))$,
8 $\forall x\, num\, x \rightarrow (\exists y\, num\, y \wedge x < y)$,
9 $\forall x\, num\, x \rightarrow (\exists y\, num\, y \wedge y < x)$,
10 $\forall \bar{x}\, \neg num\, f\bar{x}$,
11 $\forall x \forall y\, x < y \rightarrow (num\, x \wedge num\, y)$,
12 $\exists x\, num\, x$,

where f and g are distinct function symbols taken from F, x, y, z variables, $\bar{x}$ a vector of variables x_i, $\bar{y}$ a vector of variables y_i, $\bar{z}$ a vector of distinct variables z_i and $t_i(\bar{z}, \bar{x})$ a term which begins by an element of F followed by variables taken from $\bar{x}$ or $\bar{z}$.

The schemas of axiom 1, 2 and 3 are the three schemas of the theory of finite or infinite trees: the first schema called *explosion*, the second one called of *conflict of symbols* and the third one called of *unique solution*. The axioms 4, 5,..., 9 concern the relation $<$, seen as an ordered relation *strict* 4,5, *total* 6, *dense* 7 and *without endpoints* 8,9. The axioms 10, 11 and 12 are called *typing axioms*. The formulas $num\, x$ and $\neg num\, x$ are called *typing constraints* since they link a type to x.

Let us present now four properties that hold in $\mathcal{T}_{ord}$. We will use them to show the zero-infinite-decomposability of $\mathcal{T}_{ord}$. Note that the three first properties are just simple extension of the properties 3.2.4.1, 3.2.4.2 and 3.2.4.3 yet defined in the theory T_{ord} at section 3.2.4. The first property introduces the full elimination of quantifiers of Fourier for the ordered elements. The second one shows the behavior of the negation with the relation $<$. The third one holds since the relation $<$ is dense and without endpoints and introduces the notion of zero-infinite solutions in all the models of $\mathcal{T}_{ord}$. The fourth one shows the infinity of trees in all the models of $\mathcal{T}_{ord}$ according to the axioms 1 and 2 of $\mathcal{T}_{ord}$.

Property 3.4.1.1

$$\mathcal{T}_{ord} \models \left[\exists x\, num\, x \wedge \begin{bmatrix} (\bigwedge_{i \in I} x < y_i \wedge num\, y_i) \wedge \\ (\bigwedge_{j \in J} z_j < x \wedge num\, z_j) \end{bmatrix} \right] \leftrightarrow \bigwedge_{i \in I} \bigwedge_{j \in J} (z_j < y_i \wedge num\, y_i \wedge num\, z_j).$$

Property 3.4.1.2

$$\mathcal{T}_{ord} \models \forall xy\, (\neg(x < y \wedge num\, x \wedge num\, y)) \leftrightarrow \begin{bmatrix} \neg num\, x \vee \\ \neg num\, y \vee \\ (x = y \wedge num\, x \wedge num\, y) \vee \\ (y < x \wedge num\, x \wedge num\, y) \end{bmatrix}.$$

Property 3.4.1.3 *Let M be a model of $\mathcal{T}_{ord}$, let J and K be finite sets of individuals of M and let $\varphi'(x)$ be the following M-formula*

$$num\, x \wedge \ (\bigwedge_{j \in J} j < x \wedge num\, j) \wedge (\bigwedge_{k \in K} x < k \wedge num\, k).$$

The set of the individuals i of M such that $M \models \varphi'(i)$ is infinite or empty.

Property 3.4.1.4 *Let M be a model of $\mathcal{T}_{ord}$ and f a function symbol of non-nul arity. The set of the individuals i of M, such that $M \models \neg num\, i$, and the set of the individuals i of M, such that $M \models \exists \overline{x}\, i = f\overline{x}$, are infinite.*

3.4.2 The standard model of $\mathcal{T}_{ord}$

The theory $\mathcal{T}_{ord}$ has as standard model the model A_{ord} of the *finite or infinite trees with ordered leaves*, defined as follows, starting from a set D[17] disjoint from F and together with a linear dense order relation without endpoints:

Signature of A_{ord} The signature of A_{ord} is the same than those of $\mathcal{T}_{ord}$.

Domain of A_{ord} The domain $\mathcal{A}_{ord}$ of A_{ord} contains finite or infinite trees built on $F \cup D$. Each n-ary element of F is considered as a label of arity n and each element of D is considered as a label of arity 0. Since all the function symbols of F have a non-nul arity, then each tree of the domain of this model contains leaves which belong to the set D and thus all the leaves are ordered by the linear dense order relation in D. This is why the model is called *trees with ordered leaves.*

Operations of A_{ord} To each symbol $f \in F$ of arity n we link the operation of *construction* $f^{A_{ord}} : {\mathcal{A}_{ord}}^n \to \mathcal{A}_{ord}$, where $f(a_1, \ldots, a_n)$, is the tree whose root is labeled f and whose suns are $a_1, \ldots, a_n$.

Relations of A_{ord} To the relation symbol *num* we link the set $num^{A_{ord}}$ of the trees reduced to a leaf. To the relation $<$ we link the set of couples (x, y) such that $x \in num^{A_{ord}}$, $y \in num^{A_{ord}}$ and the value of x is less than the value of y.

3.4.3 Block and solved block in $\mathcal{T}_{ord}$

Definition 3.4.3.1 *We call* block*, every conjunction α of flat formulas such that each variable x in α, has at least one occurrence in a sub-formula of α of the form $num\, x$ or $\neg num\, x$. A block α without occurrences of the symbol " $=$ " is called* relational*. A block α without occurrences of the symbol " $<$ " and where each variable has an occurrence in at least an equation of α is called* equational.

Example 3.4.3.2 *The formula*

$$x = fy \wedge y < z \wedge num\, y \wedge num z,$$

is not a block because the variable x has no typing constraint. The formula

$$x = fy \wedge y < z \wedge \neg num\, x \wedge num\, y \wedge num z,$$

is a block. The block

$$x = fy \wedge y = x \wedge \neg num\, x \wedge num\, y \wedge num z,$$

is not equational because the variable z does not occur in any equation of this block. The blocks true, false and

$$x < y \wedge y < z \wedge num\, x \wedge num\, y \wedge num z,$$

are relational blocks.

[17] For example the set of the rational numbers.

Definition 3.4.3.3 *Let α be a block and $\bar{x}$ a vector of variables. A variable u is called* reachable *in $\exists\bar{x}\alpha$ if u is a free variable in $\exists\bar{x}\alpha$, or α has a sub-formula of the form $y = t(u) \wedge \neg num\, y$ with $t(u)$ a term containing u and y a reachable variable. In the last case, the equation $y = t(u)$ is called reachable in $\exists\bar{x}\alpha$.*

According to the axioms 1 and 2 of $\mathcal{T}_{ord}$, we have the following property

Property 3.4.3.4 *Let α be a block. If all the variables of $\bar{x}$ are reachable in $\exists\bar{x}\,\alpha$ then $\mathcal{T}_{ord} \models \exists?\bar{x}\alpha$.*

Suppose that the variables of V are ordered by a linear dense order relation without endpoints denoted by $\succ$.

Definition 3.4.3.5 *A block α is called* well-typed *if it does not contain sub-formulas of the form*

- $num\, x \wedge \neg num\, x$,
- $x = f\bar{y} \wedge num\, x$, *with* $f \in F$,
- $x = y \wedge num\, x \wedge \neg num\, y$,
- $x = y \wedge \neg num\, x \wedge num\, y$,
- $x < y \wedge \neg num\, x$,
- $y < x \wedge \neg num\, x$.

Definition 3.4.3.6 *A block α is called $(\succ)$-solved in $\mathcal{T}_{ord}$ if*

1. *α is well-typed,*
2. *α does not contain sub-formulas of the form $\beta \wedge false$, where β is a formula different from true,*
3. *if $x = y$ is a sub-formula of α, then $x \succ y$,*
4. *all the left hand sides of the equations of α are distinct,*
5. *if $x < y$ is a sub-formula of α then x and y do not occur in a left hand side of an equation of α,*
6. *α does not contain sub-formulas of the form*

$$x_0 < x_1 \wedge x_1 < x_2 \wedge \dots \wedge x_{n-1} < x_n \wedge x_n < x_0.$$

Example 3.4.3.7 *Let x, y, z and w be variables such that $x \succ y \succ z \succ w$. The block*

$$x = fy \wedge y < z \wedge num\, x \wedge num\, y \wedge num z,$$

is not $(\succ)$-solved because it contains a sub-formula of the form $x = fy \wedge num\, x$. The blocks true, false and

$$x = fy \wedge y = z \wedge w < z \wedge \neg num\, x \wedge num\, y \wedge num z \wedge num w,$$

are $(\succ)$-solved.

Property 3.4.3.8 *Every block is equivalent in $\mathcal{T}_{ord}$ to a wnfv ($\succ$)-solved block.*

Proof. To show this property, we introduce the following rewriting rules which transform each block into a wnfv ($\succ$)-solved block equivalent in $\mathcal{T}_{ord}$. To apply the rule $p_1 \Longrightarrow p_2$ to the block p means to replace in p, a sub-formula p_1 by the formula p_2, by considering the connector $\wedge$ associative and commutative.

$$
\begin{array}{lll}
(1) & y = f\bar{x} \wedge num\, y & \Longrightarrow \; false, \\
(2) & x < y \wedge \neg num\, x & \Longrightarrow \; false, \\
(3) & y < x \wedge \neg num\, x & \Longrightarrow \; false, \\
(4) & x = y \wedge num\, x \wedge \neg num\, y & \Longrightarrow \; false, \\
(5) & x = y \wedge num\, y \wedge \neg num\, x & \Longrightarrow \; false, \\
(6) & num\, x \wedge \neg num\, x & \Longrightarrow \; false, \\
(7) & false \wedge \alpha & \Longrightarrow \; false, \\
(8) & x = fy_1...y_m \wedge x = gz_1...z_n & \Longrightarrow \; false, \\
(9) & x = fy_1...y_n \wedge x = fz_1...z_n & \Longrightarrow \; x = fy_1...y_n \wedge \bigwedge_{i\in 1..n} y_i = z_i, \\
(10) & x_0 < x_1 \wedge ... \wedge x_{n-1} < x_n \wedge x_n < x_0 & \Longrightarrow \; false, \\
(11) & x = x & \Longrightarrow \; true, \\
(12) & y = x & \Longrightarrow \; x = y, \\
(13) & x = y \wedge x = z & \Longrightarrow \; x = y \wedge y = z, \\
(14) & x = y \wedge x = fz_1...z_n & \Longrightarrow \; x = y \wedge y = fz_1...z_n, \\
(15) & x = y \wedge x < z & \Longrightarrow \; x = y \wedge y < z, \\
(16) & x = y \wedge z < x & \Longrightarrow \; x = y \wedge z < y,
\end{array}
$$

where f and g are distinct elements taken from F, x, y, z are variables, $\bar{x}$ is a vector of variables and α is any formula. The rules (12),...,(16) are applied only if $x \succ y$. Let us show that every repeated application of these rules on a block terminates, keeps the equivalence in $\mathcal{T}_{ord}$ and produces wnfv ($\succ$)-solved block equivalent in $\mathcal{T}_{ord}$.

Proof, first part: The application of the rules terminates. Since the variables which occur in our formulas are ordered by the relation $\succ$, we can number them by positive integers such that $x \succ y \leftrightarrow no(x) > no(y)$, where $no(x)$ is the number linked to the variable x. Let us consider the 4-tuple (n_1, n_2, n_3, n_4) where the n_i are the following non-negative integer:

- n_1 is the number of sub-formulas of the form $x = fy_1...y_n$, with $f \in F$,
- n_2 is the number of occurrences of atomic formulas.
- n_3 is the sum of the $no(x)$ for each occurrence of a variable x,
- n_4 is the number of formulas of the form $x = y$, with $y \succ x$.

to each rule, there exists a row i such that the application of this rule decreases or does not change the values of the n_j with $1 \leq j < i$, and decreases the value of n_i. The row i is equal to: 1 for the rule (1), 2 for the rules (2), ..., (7), 1 for the rules (8) and (9), 2 for the rule (10), 3 for the rule (11), 4 for the rule (12) and 3 for the rules (13), (14), (15) and (16). To each sequence of formulas obtained by finite application of our rules, we can link a series of 4-tuples (n_1, n_2, n_3, n_4) which is strictly decreasing in the lexicographic order. Since These n_i's are positive integers, they can not be negative, thus this series is finite and thus the application of the rules terminates.

Proof, second part: The rules keep the equivalence in $\mathcal{T}_{ord}$. The rule (1) keeps the equivalence in $\mathcal{T}_{ord}$ according to axiom 10 and the properties of the equality. The rules (2) and (3) keep the equivalence in $\mathcal{T}_{ord}$ according to axiom 11. The rules (4) and (5) keep the equivalence in $\mathcal{T}_{ord}$ according to the properties of the equality. The rules (6) and (7) are evident in $\mathcal{T}_{ord}$. The rule (8) keeps the equivalence in $\mathcal{T}_{ord}$ according to axiom 2. The rule (9) keeps the equivalence in $\mathcal{T}_{ord}$ according to axiom 1. The rule (10) keeps the equivalence in $\mathcal{T}_{ord}$ according to axioms 4 and 5. The rules (11),...,(16) keep the equivalence in $\mathcal{T}_{ord}$ according to the properties of the equality.

Proof, third part: The application of the rules terminates by a $(\succ)$-solved block. Suppose that the application of the rules on a block ϕ terminates by a formula φ which is not a $(\succ)$-solved block. According to Definition 3.4.3.6, either φ is not a block, or φ is a block and one of the six conditions of Definition 3.4.3.6 does not hold. If φ is not a block, then there exists a variable x of φ such that $num\, x$ or $\neg num\, x$ is not a sub-formula of φ. Since each rule produces a wnfv conjunction of atomic formulas starting from a conjunction of atomic formulas and since ϕ is a block, and the only rule which removes the typing constraints is the rule 7 which can not be applied any more according to our suppositions, then φ is the formula *false*, which contradicts our supposition (φ is not a block). Suppose now that φ is a block and at least one of the six conditions of Definition 3.4.3.6 does not hold. According to which conditions 1, 2, 3, 4, 5, 6 does not hold, one at least of the following rules can still be applied (1),(2),(3),(4),(5),(6) or (7) or (11),(12) or (8),(9),(13),(14) or (15),(16) or (10), which contradicts our supposition. □

According to axiom 3 of $\mathcal{T}_{ord}$, we have the following property

Property 3.4.3.9 *Let α be an equational $(\succ)$-solved block different from the formula false. Let $\bar{x}$ be the set of the variables which occur in a left hand side of an equation of α. Let α^* be the conjunction of typing constraints of the variables of α which do not belong to $\bar{x}$. We have*

$$\mathcal{T}_{ord} \models \alpha^* \rightarrow \exists! \bar{x}\, \alpha.$$

Example 3.4.3.10 *Let x, y and z be variables such that $x \succ y \succ z$ and let α be the block $x = fy \wedge y = z \wedge \neg num\, x \wedge num\, y \wedge numz$. We have*

$$\mathcal{T}_{ord} \models num\, z \rightarrow (\exists! xy\, x = fy \wedge y = z \wedge \neg num\, x \wedge num\, y \wedge numz).$$

3.4.4 $\mathcal{T}_{ord}$ is zero-infinite-decomposable

Theorem 3.4.4.1 *The theory $\mathcal{T}_{ord}$ is zero-infinite-decomposable.*

Proof. Let us show that $\mathcal{T}_{ord}$ satisfies the conditions of Definition 3.2.1.1.

Choice of the sets $\Psi(u)$, A, A' and A''

Let BR be the set of the blocks. The sets $\Psi(u)$, A, A' and A'' are chosen as follows:

- $\Psi(u)$ contains the set of formulas of the form $\exists \bar{y}\, u = f\bar{y}$ with $f \in F$,
- A is the set BR,
- A' is the set of formulas of the form $\exists \bar{x}' \alpha'$, where
  - α' is an equational $(\succ)$-solved block different from the formula *false* and where the order $\succ$ is such that all the variables of $\bar{x}'$ are greater than the free variables of $\exists \bar{x}' \alpha'$,
  - all the equations of α' and all the variables of $\bar{x}'$ are reachable in $\exists \bar{x}' \alpha'$
- A'' is the set of the $(\succ)$-solved relational blocks which are different from the formula *false*.

It is clear that BR is closed for the conjunction, A' contains formulas of the form $\exists \bar{x}' \alpha'$ with $\alpha' \in BR$ and A'' is a sub-set of BR.

Let us show that $\mathcal{T}_{ord}$ satisfies the five conditions of Definition 3.2.1.1.

$\mathcal{T}_{ord}$ satisfies the first condition of Definition 3.2.1.1

Let us show that every formula of the form $\exists\bar{x}\,\alpha\wedge\psi$, with $\alpha\in A$ and ψ any formula, is equivalent in $\mathcal{T}_{ord}$ to a formula, wnfv, of the form

$$\exists\bar{x}'\,\alpha'\wedge(\exists\bar{x}''\,\alpha''\wedge(\exists\bar{x}'''\,\alpha'''\wedge\psi))), \tag{3.30}$$

with $\exists\bar{x}'\alpha'\in A'$, $\alpha''\in A''$, $\alpha'''\in A$ and $\mathcal{T}_{ord}\models\forall\bar{x}''\,\alpha''\rightarrow\exists!\bar{x}'''\alpha'''$.

Let us choose the order $\succ$ such that all the variables of $\bar{x}$ are greater than the free variables of $\exists\bar{x}\alpha$. Let β be a ($\succ$)-solved formula equivalent to α in $\mathcal{T}_{ord}$, (β exists according Property 3.4.3.8). Let X be the set of the variables of $\bar{x}$. Let Y_{ac} be the set of the reachable variables in $\exists\bar{x}\beta$. Let mbg be the set of the variables which occur in a left hand side of an equation of β. If β is equivalent to *false* in $\mathcal{T}_{ord}$, then the formula $\exists\bar{x}\,\alpha\wedge\psi$ is equivalent in $\mathcal{T}_{ord}$ to a decomposed formula of the form

$$\exists\varepsilon\, \mathit{false}\wedge(\exists\varepsilon\, \mathit{true}\wedge(\exists\varepsilon\, \mathit{true}\wedge\psi)).$$

Else, β is a ($\succ$)-solved conjunction of flat equations and relations. The formula $\exists\bar{x}\,\alpha\wedge\psi$ is equivalent in $\mathcal{T}_{ord}$ to a decomposed formula of the form (3.30) where:
- $\bar{x}'$ contains the elements of $X\cap Y_{ac}$,
- $\bar{x}''$ contains the elements of $(X-Y_{ac})-mbg$.
- $\bar{x}'''$ contains the elements of $(X-Y_{ac})\cap mbg$.
- α' is of the form $\alpha'_1\wedge\alpha'_2$ where α'_1 is the conjunction of all the reachable equations in $\exists\bar{x}\beta$, and where α'_2 is the conjunction of all the sub-formulas of β of the form $num\,x$ or $\neg num\,x$ with x having at least an occurrence in α'_1.
- α'' is of the form $\alpha''_1\wedge\alpha''_2$ where α''_1 is the conjunction of all the sub-formulas of β of the form $num\,x$ or $\neg num\,x$ with $x\notin\bar{x}'''$, and where α''_2 is the conjunction of all the sub-formulas of β of the form $x<y$.
- α''' is of the form $\alpha'''_1\wedge\alpha'''_2$ where α'''_1 is the conjunction of all the equations which are not reachable in $\exists\bar{x}\beta$, and where α'''_2 is the conjunction of all the sub-formulas of β of the form $num\,x$ or $\neg num\,x$ with x having at least an occurrence in α'''_1.

According to our construction, it is clear that $\exists\bar{x}'\alpha'\in A'$, $\alpha''\in A''$ and $\alpha'''\in A$. Moreover, according to Property 3.4.3.9 we have $\mathcal{T}_{ord}\models\forall\bar{x}''\alpha''\rightarrow\exists!\bar{x}'''\alpha'''$. Let us show now that (3.30) and $\exists\bar{x}\alpha\wedge\psi$ are equivalents in $\mathcal{T}_{ord}$. Let X', X'' and X''' be the sets of the variables of the vectors[18] of $\bar{x}'$, $\bar{x}''$ and $\bar{x}'''$. If β is the formula *false* then the equivalence of the decomposition is evident. Else, β is a ($\succ$)-solved block which does not contain the sub-formula *false*. Thus, according to our construction we have: $X=X'\cup X''\cup X'''$, $X'\cap X''=\emptyset$, $X'\cap X'''=\emptyset$, $X''\cap X'''=\emptyset$, for all $x''_i\in X''$ we have $x''_i\notin var(\alpha')$ and for all $x'''_i\in X'''$ we have $x'''_i\notin var(\alpha'\wedge\alpha'')$. These properties come from the definition of ($\succ$)-solved block and the order $\succ$ which has been chosen such that the quantified variables of $\exists\bar{x}\,\alpha$ are greater than the free variables of $\exists\bar{x}\,\alpha$. On the other hand, each equation and each relation of β occurs in $\alpha'\wedge\alpha''\wedge\alpha'''$ and each equation and each relation of $\alpha'\wedge\alpha''\wedge\alpha'''$ occurs in β and thus $T_{ord}\models\beta\leftrightarrow(\alpha'\wedge\alpha''\wedge\alpha''')$. We have shown that the quantifications are coherent and the equivalence $\mathcal{T}_{ord}\models\beta\leftrightarrow\alpha'\wedge\alpha''\wedge\alpha'''$ holds. According to Property 3.4.3.8, we have $\mathcal{T}_{ord}\models\alpha\leftrightarrow\beta$ and thus the decomposition keeps the equivalence in $\mathcal{T}_{ord}$.

$\mathcal{T}_{ord}$ satisfies the second condition of Definition 3.2.1.1

Let us show that if $\exists\bar{x}'\alpha'\in A'$ then $\mathcal{T}_{ord}\models\exists?\bar{x}'\alpha'$. Since $\exists\bar{x}'\alpha'\in A'$ and according to the choice of the set A', all the variables of $\bar{x}'$ are reachable in $\exists\bar{x}'\alpha'$. Using Property 3.4.3.4 we get

[18]Of course, if $\bar{x}=\varepsilon$ then $X=\emptyset$

$\mathcal{T}_{ord} \models \exists ?\bar{x}'\alpha'$.

Let us show now that if y is a free variable in $\exists \bar{x}'\alpha'$ then $\mathcal{T}_{ord} \models \exists ? y\bar{x}'\,\alpha'$, or there exists $\psi(u) \in \Psi(u)$ with $\mathcal{T}_{ord} \models \forall y\,(\exists \bar{x}'\,\alpha') \rightarrow \psi(y)$. Let y be a free variable of $\exists \bar{x}'\alpha'$. Three cases arise:

Either, y occurs in a sub-formula of α' of the form $y = t(\bar{x}', \bar{z}', y)$, where $\bar{z}'$ is the set of the free variables of $\exists \bar{x}'\alpha'$, from which we have removed the variable y, and where $t(\bar{x}', \bar{z}', y)$ is a term which begins by an element of F followed by variables taken from $\bar{x}'$ or $\bar{z}'$ or $\{y\}$. In this case, the formula $\exists \bar{x}'\alpha'$ implies in $\mathcal{T}_{ord}$ the formula $\exists \bar{x}'\, y = t(\bar{x}', \bar{z}', y)$, which implies in $\mathcal{T}_{ord}$ the formula $\exists \bar{x}'\bar{z}'w\, y = t(\bar{x}', \bar{z}', w)$, where $y = t(\bar{x}', \bar{z}', w)$ is the formula $y = t(\bar{x}', \bar{z}', y)$ in which we have replaced every free occurrence of y in the term $t(\bar{x}', \bar{z}', y)$ by the variable w. According to the choice of the set $\Psi(u)$, the formula $\exists \bar{x}'\bar{z}'w\, u = t(\bar{x}', \bar{z}', w)$ belongs to $\Psi(u)$.

Or, y occurs in a sub-formula of α', of the form $y = z$. According to the choice of the set A', the order $\succ$ is such that the variables of $\bar{x}'$ are greater than the free variables of $\exists \bar{x}'\alpha'$. On the other hand, according to Definition 3.4.3.6 of the $(\succ)$-solved formulas, we have $y \succ z$. Thus, z is a free variable in $\exists \bar{x}'\alpha'$. Since y does not occur in another left hand side of an equation of α' (because α' is $(\succ)$-solved), and since all the variables of $\bar{x}$ are reachable in $\exists \bar{x}'\alpha'$, then all the variables of $\bar{x}'$ remain reachable in $\exists \bar{x}' y\,\alpha'$ and for each value of z there exists at most a value for y. According to Property 3.4.3.4 we have $\mathcal{T}_{ord} \models \exists ?\bar{x}' y\,\alpha'$.

Or, y occurs only in right hand sides of the equations of α'. According to the choice of the set A', all the variables of $\bar{x}'$ and all the equations of α' are reachable in $\exists \bar{x}'\alpha'$. Since y does not occur in left hand sides of equations of α', then the variable y as well as the variables of $\bar{x}'$ are reachable in $\exists \bar{x}' y\,\alpha'$. Using Property 3.4.3.4 we have $\mathcal{T}_{ord} \models \exists ?\bar{x}' y\,\alpha'$.

In all the cases, $\mathcal{T}_{ord}$ satisfies the second condition of Definition 3.2.1.1.

$\mathcal{T}_{ord}$ satisfies the third condition of Definition 3.2.1.1

Let us show that $\mathcal{T}_{ord}$ satisfies the three points of the third condition of Definition 3.2.1.1.

$\mathcal{T}_{ord}$ satisfies the first point of the third condition of Definition 3.2.1.1

Let us show first that if $\alpha'' \in A''$ then the formula $\neg\alpha''$ is equivalent in $\mathcal{T}_{ord}$ to a disjunction of elements of A, i.e. to a disjunction of blocks. Let α'' be a formula of A''. According to the choice of the set A'', α'' is of the form

$$\begin{bmatrix} (\bigwedge_{\ell \in L} num\, z_\ell) \wedge (\bigwedge_{k \in K} \neg num\, v_k) \wedge \\ \bigwedge_{i \in I} \bigwedge_{j \in J_i} (x_i < y_{ij} \wedge num\, x_i \wedge num\, y_{ij}) \end{bmatrix}.$$

Thus, the formula $\neg\alpha''$ is equivalent in $\mathcal{T}_{ord}$ to

$$\begin{bmatrix} (\bigvee_{\ell \in L} \neg num\, z_\ell) \vee (\bigvee_{k \in K} \neg\neg num\, v_k) \vee \\ \bigvee_{i \in I} \bigvee_{j \in J_i} \neg(x_i < y_{ij} \wedge num\, x_i \wedge num\, y_{ij}) \end{bmatrix},$$

which according to Property 3.4.1.2 is equivalent in $\mathcal{T}_{ord}$ to

$$\begin{bmatrix} (\bigvee_{\ell \in L} \neg num\, z_\ell) \vee (\bigvee_{k \in K} num v_k) \vee \\ \bigvee_{i \in I} \bigvee_{j \in J_i} \begin{bmatrix} (\neg num\, y_{ij}) \vee \\ (\neg num\, x_i) \vee \\ (y_{ij} < x_i \wedge num\, x_i \wedge num\, y_{ij}) \vee \\ (x_i = y_{ij} \wedge num\, x_i \wedge num\, y_{ij})) \end{bmatrix} \end{bmatrix},$$

which is a disjunction of blocks.

$\mathcal{T}_{ord}$ satisfies the second point of the third condition of Definition 3.2.1.1

Let us show now that if $\alpha'' \in A''$ then, for all variable x'', the formula $\exists x''\alpha''$ is equivalent in $\mathcal{T}_{ord}$ to an element of A''. Let α'' be a formula of A'', three cases arise:

Either, x'' has no occurrences in α''. Thus, the formula $\exists x''\,\alpha''$ is equivalent in $\mathcal{T}_{ord}$ to α'' which belongs to A''.

Or, the formula $\exists x''\,\alpha''$ is of the form $\exists x''\,\alpha''_1 \wedge \neg num\, x''$ with $\alpha''_1 \in A''$ and x'' has no occurrences in α''_1. Thus, it is equivalent in $\mathcal{T}_{ord}$ to $\alpha''_1 \wedge (\exists x''\,\neg num\, x'')$, which according to Property 3.4.1.4, is equivalent in $\mathcal{T}_{ord}$ to α''_1, which belongs to A''.

Or, the formula $\exists x''\,\alpha''$ is of the form

$$\exists x''\,\alpha''_1 \wedge num\, x'' \wedge \begin{bmatrix} (\bigwedge_{i\in I} x'' < y_i \wedge num\, y_i)\wedge \\ (\bigwedge_{j\in J} z_j < x'' \wedge num\, z_j), \end{bmatrix},$$

with $\alpha''_1 \in A''$ and x'' has no occurrences in α''_1. Thus, it is equivalent in $\mathcal{T}_{ord}$ to

$$\alpha''_1 \wedge (\exists x''\, num\, x'' \wedge \begin{bmatrix} (\bigwedge_{i\in I} x'' < y_i \wedge num\, y_i)\wedge \\ (\bigwedge_{j\in J} z_j < x'' \wedge num\, z_j), \end{bmatrix}),$$

which according to Property 3.4.1.1, is equivalent in $\mathcal{T}_{ord}$ to

$$\alpha''_1 \wedge \bigwedge_{i\in I} \bigwedge_{j\in J} (z_j < y_i \wedge num\, y_i \wedge num\, z_j),$$

which belongs to A''.

$\mathcal{T}_{ord}$ satisfies the third point of the third condition of Definition 3.2.1.1

Let us show that for each variable x, we have $\mathcal{T}_{ord} \models \exists_{o\,\infty}^{\Psi(u)} x\,\varphi(x)$. Let M be a model of $\mathcal{T}_{ord}$ and $\exists x\,\varphi'(x)$ an instantiation of $\exists \bar{x}\,\varphi(x)$ by individuals of M such that $M \models \exists x\,\varphi'(x)$. Let us show that there exists an infinite set of individuals i of M which satisfy

$$M \models \varphi'(i) \wedge \neg\psi_1(i) \wedge \cdots \wedge \neg\psi_n(i),$$

with all the $\psi_j(u) \in \Psi(u)$. This condition can be replaced by the following stronger condition

$$M \models \begin{pmatrix} num\, i \vee \\ \psi_{n+1}(i) \end{pmatrix} \wedge \varphi'(i) \wedge \neg\psi_1(i) \wedge \cdot\cdot \wedge\neg\psi_n(i),$$

where $\psi_{n+1}(u)$ belongs to $\Psi(u)$ and has been chosen distinct from $\psi_1(u), \ldots, \psi_n(u)$, because the set F of the function symbols in infinite. Since for all k between 1 and n, we have

$$\mathcal{T}_{ord} \models num\, x \to \neg\psi_k(x),\ (axiom\, 10)$$

and

$$\mathcal{T}_{ord} \models \psi_{n+1}(x) \to \neg\psi_k(x),\ (axiom\, 2)$$

then the preceding condition is simplified to

$$M \models (num\, i \wedge \varphi'(i)) \vee (\psi_{n+1}(i) \wedge \varphi'(i)),$$

and thus knowing $M \models \exists x\,\varphi'(x)$, it is enough to show that there exists an infinite set of individuals i of M such that

$$M \models num\, i \wedge \varphi'(i) \ \text{ or } \ M \models \psi_{n+1}(i) \wedge \varphi'(i). \tag{3.31}$$

Two cases arise:

Either, the formula $num\, x$ occurs in $\varphi'(x)$. Since $\varphi'(x)$ is an instantiation of a solved relational block and $M \models \exists x\, \varphi'(x)$, the formula $num\, x \wedge \varphi'(x)$ is equivalent in M to an M-formula of the form

$$num\, x \wedge \ (\bigwedge_{j \in J} j < x \wedge num\, j) \wedge (\bigwedge_{k \in K} x < k \wedge num\, k).$$

according to Property 3.4.1.3 and since $M \models \exists x\, num\, x \wedge \varphi'(x)$, then there exists an infinity of individuals i of M such that $M \models num\, i \wedge \varphi'(i)$ and thus such that (3.31).

Or, the formula $num\, x$ does not occur in $\varphi'(x)$. Since $\varphi'(x)$ is an instantiation of a solved relational block and $M \models \exists x\, \varphi'(x)$, the M-formula $\psi_{n+1}(x) \wedge \varphi'(x)$ is equivalent in M to $\psi_{n+1}(x)$. According to Property 3.4.1.4, there exists an infinity of individuals i of M such that $M \models \psi_{n+1}(i)$, thus such that $M \models \psi_{n+1}(i) \wedge \varphi'(i)$ and thus such that (3.31).

In all the cases, $\mathcal{T}_{ord}$ satisfies the third condition of Definition 3.2.1.1.

$\mathcal{T}_{ord}$ satisfies the fourth condition of Definition 3.2.1.1

Let us show that every conjunction of flat formulas φ is equivalent in $\mathcal{T}_{ord}$ to a disjunction of blocks. For that, it is enough to show that every flat formula is equivalent in $\mathcal{T}_{ord}$ to a disjunction of blocks. Let φ be a flat formula. If it is of the form *true*, *false* or $num\, x$ then φ is a block. Else, the following equivalences after distribution of the $\wedge$ on the $\vee$ give the needed combinations:

$$\mathcal{T}_{ord} \models x_0 = x_1 \leftrightarrow x_0 = x_1 \wedge \begin{bmatrix} (num\, x_0 \vee \neg num\, x_0) \wedge \\ (num\, x_1 \vee \neg num\, x_1) \end{bmatrix}$$

$$\mathcal{T}_{ord} \models x_0 < x_1 \leftrightarrow x_0 < x_1 \wedge \begin{bmatrix} (num\, x_0 \vee \neg num\, x_0) \wedge \\ (num\, x_1 \vee \neg num\, x_1) \end{bmatrix}$$

$$\mathcal{T}_{ord} \models x_0 = f x_1 ... x_n \leftrightarrow \begin{bmatrix} x_0 = f x_1 ... x_n \wedge \\ \bigwedge_{i \in 0..n} (num\, x_i \vee \neg num\, x_i) \end{bmatrix}.$$

$\mathcal{T}_{ord}$ satisfies the fifth condition

Let us show that for every formula without free variables of the form $\exists \bar{x}' \alpha' \wedge \alpha''$ with $\exists \bar{x}' \alpha' \in A'$ and $\alpha'' \in A''$ we have $\bar{x} = \varepsilon$, $\alpha' \in \{true, false\}$ and $\alpha'' \in \{true, false\}$. Since the formula $\exists \bar{x}' \alpha' \wedge \alpha''$ does not contain free variables, then there exists neither reachable variables nor reachable equations in $\exists \bar{x}' \alpha' \wedge \alpha''$. Thus, according to the choice of the set A', the formula $\exists \bar{x}' \alpha'$ is of the form $\exists \varepsilon true$. On the other hand, according to the choice of the set A'', the formula *true* is the only one which belongs to A'' and has no free variables. Thus, $\mathcal{T}_{ord}$ satisfies the fifth condition of Definition 3.2.1.1.

We have shown that $\mathcal{T}_{ord}$ satisfies the five conditions of Definition 3.2.1.1. Thus, $\mathcal{T}_{ord}$ is zero-infinite-decomposable.

3.4.5 Solving first order propositions in $\mathcal{T}_{ord}$

Let us solve in $\mathcal{T}_{ord}$ the following proposition φ_1:

$$\neg(\exists y \forall x v \exists z\, y = fz \wedge y = fx \wedge v < z \wedge \neg num\, y \wedge num\, z). \tag{3.32}$$

Note that the variables x and v are not typed. This formula is true in $\mathcal{T}_{ord}$ because the variables x and v are universally quantified which contradicts the fact that $v < z$. Let us use our algorithm to solve this proposition. Let us transform first the formula (3.32) into a normalized formula equivalent in $\mathcal{T}_{ord}$ using Property 3.3.1.2. Thus, the formula (3.32) is equivalent in $\mathcal{T}_{ord}$ to the following normalized formula

$$\neg\left[\begin{array}{l}\exists y\, true \wedge \\ \neg\left[\begin{array}{l}\exists xv\, true \wedge \\ \neg(\exists z\, y = fz \wedge y = fx \wedge v < z \wedge \neg num\, y \wedge num\, z)\end{array}\right]\end{array}\right].$$

Let us transform now this formula into a conjunction of working formulas using Property 3.3.2.2. The preceding formula is equivalent in $\mathcal{T}_{ord}$ to the following working formula

$$\neg\left[\begin{array}{l}\exists y\, true \wedge \\ \neg\left[\begin{array}{l}\exists xv\, true \wedge \\ \neg(\exists z_1\, y = fz_1 \wedge y = fx \wedge v < z_1 \wedge \neg num\, x \wedge \neg num\, y \wedge num\, z_1 \wedge \neg num\, v) \\ \wedge \\ \neg(\exists z_2\, y = fz_2 \wedge y = fx \wedge v < z_2 \wedge \neg num\, x \wedge \neg num\, y \wedge num\, z_2 \wedge num\, v) \\ \wedge \\ \neg(\exists z_3\, y = fz_3 \wedge y = fx \wedge v < z_3 \wedge num\, x \wedge \neg num\, y \wedge num\, z_3 \wedge \neg num\, v) \\ \wedge \\ \neg(\exists z_4\, y = fz_4 \wedge y = fx \wedge v < z_4 \wedge num\, x \wedge \neg num\, y \wedge num\, z_4 \wedge num\, v)\end{array}\right]\end{array}\right]. \tag{3.33}$$

The three blocks:

- $\exists z_1\, y = fz_1 \wedge y = fx \wedge v < z_1 \wedge \neg num\, x \wedge \neg num\, y \wedge num\, z_1 \wedge \neg num\, v$,
- $\exists z_2\, y = fz_2 \wedge y = fx \wedge v < z_2 \wedge \neg num\, x \wedge \neg num\, y \wedge num\, z_2 \wedge num\, v$,
- $\exists z_3\, y = fz_3 \wedge y = fx \wedge v < z_3 \wedge num\, x \wedge \neg num\, y \wedge num\, z_3 \wedge \neg num\, v$,

are equivalents in $\mathcal{T}_{ord}$ to the $(\succ)$-solved block *false*. We can then apply the rule (5) with $I = \emptyset$ three times on the formula (3.33). Thus, it is equivalent in $\mathcal{T}_{ord}$ to

$$\neg\left[\begin{array}{l}\exists y\, true \wedge \\ \neg\left[\begin{array}{l}\exists xv\, true \wedge \\ \neg(\exists z_1\, false) \\ \wedge \\ \neg(\exists z_2\, false) \\ \wedge \\ \neg(\exists z_3\, false) \\ \wedge \\ \neg(\exists z_4\, y = fz_4 \wedge y = fx \wedge v < z_4 \wedge num\, x \wedge \neg num\, y \wedge num\, z_4 \wedge num\, v)\end{array}\right]\end{array}\right],$$

which after application of the rule (2) three times is equivalent in $\mathcal{T}_{ord}$ to

$$\neg\left[\begin{array}{l}\exists y\, true \wedge \\ \neg\left[\begin{array}{l}\exists xv\, true \wedge \\ \neg(\exists z_4\, y = fz_4 \wedge y = fx \wedge v < z_4 \wedge num\, x \wedge \neg num\, y \wedge num\, z_4 \wedge num\, v)\end{array}\right]\end{array}\right]. \tag{3.34}$$

Let us now treat the sub-formula

$$\exists z_4\, y = fz_4 \wedge y = fx \wedge v < z_4 \wedge num\, x \wedge \neg num\, y \wedge num\, z_4 \wedge num\, v. \tag{3.35}$$

Let us choose the order $\succ$ such that $z_4 \succ x \succ y \succ v$. According to Property 3.4.3.8, the block

$$y = fz_4 \wedge y = fx \wedge v < z_4 \wedge num\, x \wedge \neg num\, y \wedge num\, z_4 \wedge num\, v,$$

is equivalent in $\mathcal{T}_{ord}$ to the ($\succ$)-solved block

$$y = fz_4 \wedge z_4 = x \wedge v < x \wedge num\, x \wedge \neg num\, y \wedge num\, z_4 \wedge num\, v.$$

thus, the formula (3.35) is equivalent in $\mathcal{T}_{ord}$ to the decomposed formula

$$\begin{bmatrix} \exists z_4\, y = fz_4 \wedge z_4 = x \wedge num\, x \wedge \neg num\, y \wedge num\, z_4 \wedge \\ (\exists \varepsilon\, v < x \wedge num\, x \wedge num\, v \wedge \\ (\exists \varepsilon\, true)) \end{bmatrix}.$$

Thus, we can apply the rule (5) on the formula (3.34) with $I = \emptyset$. The formula (3.34) is equivalent in $\mathcal{T}_{ord}$ to

$$\neg \begin{bmatrix} \exists y\, true \wedge \\ \neg \begin{bmatrix} \exists xv\, true \wedge \\ \neg \begin{bmatrix} \exists z_4\, y = fz_4 \wedge z_4 = x \wedge num\, x \wedge \neg num\, y \wedge num\, z_4 \wedge \\ (v < x \wedge num\, x \wedge num\, v) \end{bmatrix} \end{bmatrix} \end{bmatrix}.$$

Note that the sub-formula $(v < x \wedge num\, x \wedge num\, v)$ represents the formula α''_* in the rule (5). Since

- $(\exists z_4\, y = fz_4 \wedge z_4 = x \wedge num\, x \wedge \neg num\, y \wedge num\, z_4) \in A'$,
- $(v < x \wedge num\, x \wedge num\, v) \in A''$,
- $\mathcal{T}_{ord} \models \neg(v < x \wedge num\, x \wedge num\, v) \leftrightarrow \begin{bmatrix} (x < v \wedge num\, x \wedge num\, v) \vee \\ (v = x \wedge num\, x \wedge num\, v) \vee \\ \neg num\, x \vee \\ \neg num\, v \end{bmatrix}$

then we can apply the rule (4). The preceding formula is thus equivalent in T_{ord} to

$$\neg \begin{bmatrix} \exists y\, true \wedge \\ \neg \begin{bmatrix} \exists xv\, true \wedge \\ \neg(\exists z_4\, y = fz_4 \wedge z_4 = x \wedge num\, x \wedge \neg num\, y \wedge num\, z_4) \end{bmatrix} \\ \wedge \\ \neg \left[\exists x_1 v_1 z_5\, y = fz_5 \wedge z_5 = x_1 \wedge \neg num\, y \wedge num\, z_5 \wedge x_1 < v_1 \wedge num\, x_1 \wedge num\, v_1 \right] \\ \wedge \\ \neg \left[\exists x_2 v_2 z_6\, y = fz_6 \wedge z_6 = x_2 \wedge \neg num\, y \wedge num\, z_6 \wedge v_2 = x_2 \wedge num\, x_2 \wedge num\, v_2 \right] \\ \wedge \\ \neg \left[\exists x_3 v_3 z_7\, y = fz_7 \wedge z_7 = x_3 \wedge num\, x_3 \wedge \neg num\, y \wedge num\, z_7 \wedge \neg num\, v_3 \right] \\ \wedge \\ \neg \left[\exists x_4 v_4 z_8, y = fz_8 \wedge z_8 = x_4 \wedge num\, x_4 \wedge \neg num\, y \wedge num\, z_8 \wedge \neg num\, x_4 \right] \end{bmatrix},$$

which, since the last sub-working-formula is equivalent to the ($\succ$)-solved formula *false* (because we have $num\, x_4 \wedge \neg num\, x_4$) and after application of the rules (5) and (2) with $I = \emptyset$, is equivalent in $\mathcal{T}_{ord}$ to

$$\neg\left[\begin{array}{l}\exists y\, true \wedge \\ \neg\left[\begin{array}{l}\exists xv\, true \wedge \\ \neg(\exists z_4\, y = fz_4 \wedge z_4 = x \wedge num\, x \wedge \neg num\, y \wedge num\, z_4)\end{array}\right] \\ \wedge \\ \neg\left[\ \exists x_1 v_1 z_5\, y = fz_5 \wedge z_5 = x_1 \wedge \neg num\, y \wedge num\, z_5 \wedge x_1 < v_1 \wedge num\, x_1 \wedge num\, v_1\ \right] \\ \wedge \\ \neg\left[\ \exists x_2 v_2 z_6\, y = fz_6 \wedge z_6 = x_2 \wedge \neg num\, y \wedge num\, z_6 \wedge v_2 = x_2 \wedge num\, x_2 \wedge num\, v_2\ \right] \\ \wedge \\ \neg\left[\ \exists x_3 v_3 z_7\, y = fz_5 \wedge z_7 = x_3 \wedge num\, x_3 \wedge \neg num\, y \wedge num\, z_7 \wedge \neg num\, v_3\ \right]\end{array}\right]. \tag{3.36}$$

Since the formula $\exists xv\, true$ is equivalent to the decomposed formula $\exists \varepsilon\, true \wedge (\exists xv\, true \wedge (\exists \varepsilon\, true))$ and $(\exists z_4\, y = fz_4 \wedge z_4 = x \wedge num\, x \wedge \neg num\, y \wedge num\, z_4) \in A'$ then we can apply the rule (5) with $I = \neg(\exists z_4\, y = fz_4 \wedge z_4 = x \wedge num\, x \wedge \neg num\, y \wedge num\, z_4)$. The formula (3.36) is thus equivalent in $\mathcal{T}_{ord}$ to

$$\neg\left[\begin{array}{l}\exists y\, true \wedge \\ \neg\left[\ \exists \varepsilon\, true\ \right] \\ \wedge \\ \neg\left[\ \exists x_1 v_1 z_5\, y = fz_5 \wedge z_5 = x_1 \wedge \neg num\, y \wedge num\, z_5 \wedge x_1 < v_1 \wedge num\, x_1 \wedge num\, v_1\ \right] \\ \wedge \\ \neg\left[\ \exists x_2 v_2 z_6\, y = fz_6 \wedge z_6 = x_2 \wedge \neg num\, y \wedge num\, z_6 \wedge v_2 = x_2 \wedge num\, x_2 \wedge num\, v_2\ \right] \\ \wedge \\ \neg\left[\ \exists x_3 v_3 z_7\, y = fz_7 \wedge z_7 = x_3 \wedge num\, x_3 \wedge \neg num\, y \wedge num\, z_7 \wedge \neg num\, v_3\ \right]\end{array}\right], \tag{3.37}$$

which after application of the rule (1) is equivalent to *true* in $\mathcal{T}_{ord}$. Thus, φ_1 is true in $\mathcal{T}_{ord}$.

3.5 Discussion and partial conclusion

We have presented in this chapter the class of the zero-infinite-decomposable theories, which is an extension of the infinite-decomposable theories, by replacing the infinite quantifier by the zero-infinite quantifier. We have also given a property that links the infinite-decomposable theories with the zero-infinite-decomposable theories and have shown that the theories T_{ord} and $\mathcal{T}_{ord}$ are zero-infinite-decomposable and thus complete.

The decision procedure defined in this chapter contains a new rule comparing with those of the infinite-decomposable theories, which associates a particular treatment to the formulas of the form $\neg(\exists \bar{x}'\, \alpha' \wedge \alpha'')$ with $\exists \bar{x}'\, \alpha' \in A'$ and $\alpha'' \in A''$. This decision procedure as well as those defined in Chapter 2, is a proposition decision algorithm which for every proposition gives either *true* or *false*. It can also be applied to formulas having at least one free variable and gives in this case a conjunction ϕ of solved formulas easily transformable into a boolean combination of basic formulas. Unfortunately, it does not warrant that the formula ϕ is neither true nor false if it contains at least one free variables and is not able to present the solutions of the free variables in a clear and explicit way.

On the other hand, by introducing the theory $\mathcal{T}_{ord}$ we have felt the first intuitions of the combination of any first order theory T with the theory of finite or infinite trees. It will be very interesting to define an automatic way to combine any first order theory T with the theory of finite or infinite trees and show the completeness of this new hybrid theory using the zero-infinite-decomposable theories. This will be our goal in Chapter 4 !

Chapter 4

Extension into trees T^* of a first order theory T

Contents

We present in this chapter an automatic way to combine any first order theory T with the theory of finite or infinite trees. One of the major difficulties in this combination resides in the fact that the theory T and the theory of finite or infinite trees can have non-disjoint signatures, i.e. there exists at least a function symbol having two completely different behaviors whether we are in the theory of finite or infinite trees or in the theory T. Thus, we must first find a semantic meaning to this combination, then give a harmonious axiomatization of this melange. For that, we define semantically this combination as an extension into trees of the elements of the models of the theory T. Then, the axiomatization of the extension into trees of T, denoted by T^*, is made essentially from the three axioms of Michael Maher of the theory of finite or infinite trees [33] and from the axiomatization of the theory T by introducing typing constraints. We also present the standard model M^* of the theory T^* using the standard model M of T. To show the completeness of the theory T^*, we introduce a new class of theories that we call *flexible* and show that if T is a flexible theory, then its extension into trees, i.e. T^*, is zero-infinite-decomposable and thus complete. The flexible theories are theories having elegant properties which enable us to handle easily the first order formulas. We end this chapter by an application to the extension into trees T^*_{ad} of the theory T_{ad} of ordered additive rational numbers. We show that T_{ad} is flexible and thus T^*_{ad} is complete. Note that the results presented in this chapter have been published in [15], [20].

4.1 Extension into trees T^* of T

Since we will handle theories with different signatures S and S^*, then we will add to the words: equation, relation, term, formula, model and theory the prefixes S or S^* whether we are in the signature S or S^*. An S-term for example is a term built on the signature S while an S^*-term is a term built on the signature S^*. More details are available in Chapter 1.

4.1.1 Axiomatization of T^*

Let us recall first the three axioms of the theory of finite or infinite trees. Let S be a signature containing only an infinite set F of function symbols. The axiomatization of the S-theory of finite or infinite trees consists in the set of the following S-propositions:

$$
\begin{array}{ll}
1 & \forall\bar{x}\forall\bar{y}\, f\bar{x} = f\bar{y} \rightarrow \bigwedge_i x_i = y_i,\\
2 & \forall\bar{x}\forall\bar{y}\, \neg f\bar{x} = g\bar{y},\\
3 & \forall\bar{x}\exists!\bar{z}\, \bigwedge_i z_i = t_i(\bar{z},\bar{x}),
\end{array}
$$

with f and g two distinct function symbols taken from F, x, y, z variables, $\bar{x}$ a vector of variables x_i, $\bar{y}$ a vector of variables y_i, $\bar{z}$ a vector of distinct variables z_i and $t_i(\bar{x},\bar{z})$ an S-term which begins by an element of F followed by variables taken from $\bar{x}$ or $\bar{z}$.

Let us fix now a signature S containing a set F of function symbols and a set R of relation symbols, as well as a signature S^* containing:

- an infinite set $F^* = F \cup F_A$ where F_A is an infinite set of function symbols, each one of non-nul arity, and disjoint from F.
- a set $R^* = R \cup \{p\}$ of relation symbols, containing R, as well as a relation symbol p of arity 1.

Let T be an S-theory. The extension into trees of the S-theory T is the S^*-theory T^* whose axioms are the infinite set of the following S^*-propositions, with $\bar{x}$ a vector of variables x_i and $\bar{y}$ a vector of variables y_i:

1. Explosion: for all $f \in F^*$:

$$\forall\bar{x}\forall\bar{y}\ \neg pf\bar{x} \wedge \neg pf\bar{y} \wedge f\bar{x} = f\bar{y} \rightarrow \bigwedge_i x_i = y_i$$

2. Conflict of symbols: Let f and g be two distinct function symbols taken from F^*:

$$\forall\bar{x}\forall\bar{y}\ f\bar{x} = g\bar{y} \rightarrow pf\bar{x} \wedge pg\bar{y}$$

3. Unique solution

$$\forall\bar{x}\forall\bar{y}\ (\bigwedge_i px_i) \wedge (\bigwedge_j \neg py_j) \rightarrow \exists!\bar{z} \bigwedge_k (\neg pz_i \wedge z_k = t_k(\bar{x},\bar{y},\bar{z}))$$

   where $\bar{z}$ is a vector of distinct variables z_i, $t_k(\bar{x},\bar{y},\bar{z})$ an S^*-term which begins by a function symbol $f_k \in F^*$ followed by variables taken from $\bar{x},\bar{y},\bar{z}$, moreover, if $f_k \in F$, then the S^*-term $t_k(\bar{x},\bar{y},\bar{z})$ contains at least a variable from $\bar{y}$ or $\bar{z}$

4. Relations of R: for all $r \in R$,
$$\forall \bar{x}\ r\bar{x} \rightarrow \bigwedge_i px_i$$

5. Operations of F: for all $f \in F$,
$$\forall \bar{x}\ pf\bar{x} \leftrightarrow \bigwedge_i px_i$$
(if f is 0-ary then this axiom is written pf)

6. Elements not in T: for all $f \in F^* - F$,
$$\forall \bar{x}\ \neg pf\bar{x}$$

7. Existence of an element satisfying p (only if F does not contain 0-ary function symbols):
$$\exists x\, px,$$

8. Extension into trees of the axioms of T: all axioms obtained by the following transformations of each axiom φ of T : While it is possible replace all sub-formula of φ which is of the form $\exists \bar{x}\, \psi$, but not of the form $\exists \bar{x}\, (\bigwedge px_i) \wedge \psi'$, by $\exists \bar{x}\, (\bigwedge px_i) \wedge \psi$ and every sub-formula of φ which is of the form $\forall \bar{x}\, \psi$, but not of the form $\forall \bar{x}\, (\bigwedge px_i) \rightarrow \psi'$, by $\forall \bar{x}\, (\bigwedge px_i) \rightarrow \psi$.

4.1.2 The standard model M^* of T^*

Let $M = (\mathcal{M}, \mathcal{F}, \mathcal{R})$ be an S-model of an S-theory T with $\mathcal{F}$ a set of functions in $\mathcal{M}$ subscripted by the elements of F and $\mathcal{R}$ a set of relations in $\mathcal{M}$ subscripted by the elements of R.

Let $M^* = (\mathcal{M}^*, \mathcal{F}^*, \mathcal{R}^*)$ be an S^*-model with $\mathcal{F}^*$ an infinite set of functions subscripted by the elements of F^* and containing the set $\mathcal{F}$, and $\mathcal{R}^* = \mathcal{R} \cup \{p\}$ a set of relations subscripted by the elements of R^* and containing the set $\mathcal{R}$ as well as an 1-ary relation p.

The extension into trees T^* of the S-theory T has as standard model the extension into trees of the S-model M, i.e. the S^*-model $M^* = (\mathcal{M}^*, \mathcal{F}^*, \mathcal{R}^*)$ defined as follows[19] :

Domain of M^* : The domain $\mathcal{M}^*$ is the set of the finite or infinite trees labeled by $F^* \cup \mathcal{M}$ by considering each n-ary symbol in F^* as a label of arity n and each individual of $\mathcal{M}$ as a label of arity 0 and such that each sub-tree labeled by $F \cup \mathcal{M}$ is evaluated in $\mathcal{M}$ and reduced to a leaf labeled by an element of $\mathcal{M}$. Since F^* does not contain function symbols of arity 0 then all the leaves of any tree a taken from $\mathcal{M}^*$ belong to $\mathcal{M}$. We understand now the semantic meaning of an extension into trees of any theory T which is finally nothing else a construction of trees on the individuals of each model M_i of T without creating new leaves that does not belong to $\mathcal{M}_\rangle$.

Operations of M^* : To each n-ary function symbol f in F^* is associated the application $f^{M^*} : \mathcal{M}^{*n} \rightarrow \mathcal{M}^*$ such that $f(a_1, .., a_n)$ is the result of f on $(a_1, .., a_n)$ in $\mathcal{M}$, if $f \in F$ and $a_i \in \mathcal{M}$ for all $i \in \{1, ..., n\}$, and is the tree whose root is labeled f and whose suns are $(a_1, .., a_n)$ else.

Relations of M^* : To each n-ary relation symbol r of $R^* - \{p\}$ is associated the set $r^{M^*} = r^M$. To the relation symbol p is associated the set $p^{M^*} = \mathcal{M}$.

[19] By denoting by $(f^{M^*})_{f \in F^*}$ and $(r^{M^*})_{r \in R^*}$ for $\mathcal{F}^*$ respectively $\mathcal{R}^*$.

4.1.3 Examples

Extension into trees of the empty theory

Let $S = \emptyset$ be an empty signature and T be the S-empty theory. This empty theory has as model every non-empty set without any other restrictions. Let $S^* = F^* \cup \{p\}$ be a signature such that F^* is an infinite set of function symbols, each one having a non-nul arity, and p a relation symbol of arity 1. The extension into trees of T is the S^*-theory T^* whose set of axioms is the set of the following propositions:

1. Explosion: for all $f \in F^*$:
$$\forall \bar{x} \forall \bar{y}\ \neg pf\bar{x} \wedge \neg pf\bar{y} \wedge f\bar{x} = f\bar{y} \rightarrow \bigwedge_i x_i = y_i$$

2. Conflict of symbols: Let f and g be two distinct function symbols taken from F^*:
$$\forall \bar{x} \forall \bar{y}\ f\bar{x} = g\bar{y} \rightarrow pf\bar{x} \wedge pg\bar{y}$$

3. Unique solution
$$\forall \bar{x} \forall \bar{y}\ (\bigwedge_i px_i) \wedge (\bigwedge_j \neg py_j) \rightarrow \exists! \bar{z} \bigwedge_k (\neg pz_i \wedge z_k = t_k(\bar{x}, \bar{y}, \bar{z}))$$
where $\bar{z}$ is a vector of distinct variables z_i, $t_k(\bar{x}, \bar{y}, \bar{z})$ an S^*-term which begins by a function symbol $f_k \in F^*$ followed by variables taken from $\bar{x}, \bar{y}, \bar{z}$,

4. Elements not in T: for all $f \in F^*$,
$$\forall \bar{x}\ \neg pf\bar{x}$$

5. Existence of an element satisfying p :
$$\exists x\, px.$$

We can simplify this axiomatization using Axiom 4. We will also replace the relation symbol p by $leaf$ in order to clarify the intuitions of the our axiomatization. Thus, we get the following axiomatization:

1. Explosion: for all $f \in F^*$:
$$\forall \bar{x} \forall \bar{y}\ f\bar{x} = f\bar{y} \rightarrow \bigwedge_i x_i = y_i$$

2. Conflict of symbols: Let f and g be two distinct function symbols taken from F^* :
$$\forall \bar{x} \forall \bar{y}\ f\bar{x} = g\bar{y}$$

3. Unique solution
$$\forall \bar{x} \exists! \bar{z} \bigwedge_k z_k = t_k(\bar{x}, \bar{z})$$
where $\bar{z}$ is a vector of distinct variables z_i, $t_k(\bar{x}, \bar{z})$ an S^*-term which begins by a function symbol $f_k \in F^*$ followed by variables taken from $\bar{x}$ or $\bar{z}$,

4. Elements not in T: for all $f \in F^*$,

$$\forall \bar{x}\ \neg leaf\ f\bar{x}$$

5. Existence of an element satisfying $leaf$:

$$\exists x\, leaf\ x.$$

This axiomatization is the axiomatization of the theory of finite or infinite trees of M. Maher [33], built on the set F^* and increased by the relation symbol $leaf$ of arity 1 which distinguishes the leaves from the other trees. Nevertheless, this axiomatization forces each models of T^* to contain at least a tree reduced to a leaf. This small restriction is due to the fact that according to the definition of model (see Chapter 1), each model M of the empty theory contains at least one individual. Thus, the extension M^* of the model M contains at least an individual which will be reduced to a leaf.

Extension into trees ${T_{ord}}^*$ of the linear dense order relation without endpoints T_{ord}

Let F be an empty set of function symbols and let R be a set of relation symbols containing only the relation symbol $<$ of arity 2. If t_1 and t_2 are terms, then we write $t_1 < t_2$ for $<(t_1, t_2)$. Let T_{ord} the theory of the linear dense order relation without endpoints, whose signature is $S = F \cup R$ and whose axioms are the following propositions:

1 $\forall x \neg x < x$,
2 $\forall x \forall y \forall z\, (x < y \wedge y < z) \rightarrow x < z$,
3 $\forall x \forall y\, x < y \vee x = y \vee y < x$,
4 $\forall x \forall y\, x < y \rightarrow (\exists z\, x < z \wedge z < y)$,
5 $\forall x\, \exists y\, x < y$,
6 $\forall x\, \exists y\, y < x$.

Let now F^* be an infinite set of function symbols each one of a non-nul arity and $R^* = \{<, p\}$ a set of relation symbol containing the symbol $<$ as well as the relation symbol p. Let S^* be the signature $F^* \cup R^*$. According to the transformations of axioms in Section 4.1.1, the axiomatization of the extension into trees of the theory T_{ord} is the S^*-theory ${T_{ord}}^*$ whose axioms are the following propositions:

1 $\forall \bar{x} \forall \bar{y}\ \neg pf\bar{x} \wedge \neg pf\bar{y} \wedge f\bar{x} = f\bar{y} \rightarrow \bigwedge_i x_i = y_i$
2 $\forall \bar{x} \forall \bar{y}\ f\bar{x} = g\bar{y} \rightarrow pf\bar{x} \wedge pg\bar{y}$
3 $\forall \bar{x} \forall \bar{y}\ (\bigwedge_i px_i) \wedge (\bigwedge_j \neg py_j) \rightarrow \exists! \bar{z} \bigwedge_k (\neg pz_i \wedge z_k = t_k(\bar{x}, \bar{y}, \bar{z}))$
4 $\forall x \forall y\, x < y \rightarrow (px \wedge py)$,
5 $\forall \bar{x} \neg pf\bar{x}$,
6 $\exists x\, px$,
7 $\forall x\, px \rightarrow \neg x < x$,
8 $\forall x \forall y \forall z\, px \wedge py \wedge pz \rightarrow ((x < y \wedge y < z) \rightarrow x < z)$,
9 $\forall x \forall y\, (px \wedge py) \rightarrow (x < y \vee x = y \vee y < x)$,
10 $\forall x \forall y\, (px \wedge py) \rightarrow (x < y \rightarrow (\exists z\, pz \wedge x < z \wedge z < y))$,
11 $\forall x\, px \rightarrow (\exists y\, py \wedge x < y)$,
12 $\forall x\, px \rightarrow (\exists y\, py \wedge y < x)$,

where f and g are distinct function symbols taken from F^*, x, y, z variables, $\bar{x}$ a vector of variables x_i, $\bar{y}$ a vector of variables y_i, $\bar{z}$ a vector of distinct variables z_i and $t_i(\bar{x}, \bar{y}, \bar{z})$ a term which begins by an element de F^* followed by variables taken from $\bar{x}$, $\bar{y}$ or $\bar{z}$.

According to axiom 5, and by replacing the relation symbol p by the relation symbol num, this axiomatization is simplified to

1 $\forall \bar{x} \forall \bar{y}\ f\bar{x} = f\bar{y} \rightarrow \bigwedge_i x_i = y_i$
2 $\forall \bar{x} \forall \bar{y}\ \neg(f\bar{x} = g\bar{y})$
3 $\forall \bar{x}\ \exists! \bar{z} \bigwedge_k z_k = t_k(\bar{x}, \bar{z})$
4 $\forall x \forall y\, x < y \rightarrow (num\, x \wedge num\, y)$,
5 $\forall \bar{x} \neg num\, f\bar{x}$,
6 $\exists x\, num\, x$,
7 $\forall x\, num\, x \rightarrow \neg x < x$,
8 $\forall x \forall y \forall z\, num\, x \wedge num\, y \wedge num\, z \rightarrow ((x < y \wedge y < z) \rightarrow x < z)$,
9 $\forall x \forall y\, (num\, x \wedge num\, y) \rightarrow (x < y \vee x = y \vee y < x)$,
10 $\forall x \forall y\, (num\, x \wedge num\, y) \rightarrow (x < y \rightarrow (\exists z\, num\, z \wedge x < z \wedge z < y))$,
11 $\forall x\, num\, x\ \rightarrow (\exists y\, num\, y \wedge x < y)$,
12 $\forall x\, num\, x\ \rightarrow (\exists y\, num\, y \wedge y < x)$,

where f and g are distinct function symbols taken from F^*, x, y, z variables, $\bar{x}$ a vector of variables x_i, $\bar{y}$ a vector of variables y_i, $\bar{z}$ a vector of distinct variables z_i and $t_i(\bar{z}, \bar{x})$ a term which begins by an element of F^* followed by variables taken from $\bar{x}$ or $\bar{z}$.

This axiomatization is the same than those of the construction of trees on an ordered set $\mathcal{T}_{ord}$ given in Chapter 3.

4.2 Completeness of T^*

Let us fix for all this section a signature S containing a set F of function symbols and a set R of relation symbols, as well as a signature S^* containing

- an infinite set $F^* = F \cup F_A$ where F_A is an infinite set of function symbols, each one of non-nul arity, and disjoint from F,
- a set $R^* = R \cup \{p\}$ of relation symbols, containing R as well as a relation symbol p of arity 1.

Let us fix also an S-theory T and its extension into trees T^*.

Suppose that the variables of V are ordered by a linear dense order relation without endpoints denoted by $\succ$.

4.2.1 Flexible theory

Definition 4.2.1.1 *We call* leader *of an S-equation α the greatest variable x in α, according to the order $\succ$, such that $T \models \exists! x \alpha$.*

Example 4.2.1.2 *Let x, y and z be variables such that $x \succ y \succ z$. Let us consider the theory Ra of the additive rational numbers defined in Chapter 2. The variable x is leader of the equation $2.x + 3.y = 1 + 2.z$ because x is the greatest variable and $Ra \models \exists! x\, 2.x + 3.y = 1 + 2.z$.*

Definition 4.2.1.3 *A conjunction of S-atomic formulas α is called* formated *in T if*

- *α does not contain sub-formulas of the form $f_1 = f_2$ or $r f_1 ... f_n$ or $y = x$, where all the f_i are 0-ary function symbols taken from F, $r \in R$ and $x \succ y$,*

- *each S-equation of α has a distinct leader which has no occurrences in other S-equations or S-relations of α,*
- *if α' is the conjunction of all the S-equations of α then for all $x \in var(\alpha')$ we have $T \models \exists ?x\, \alpha'$.*

Let us introduce now a class of theories T_i whose properties will enable us to show the zero-infinite-decomposability of their extension into trees.

Definition 4.2.1.4 *The theory T is called* flexible *if for each conjunction α of S-equations and for each conjunction β of S-relations:*

1. *$\alpha \wedge \beta$ is equivalent in T to a formated conjunction of atomic formulas wnfv,*
2. *the S-formula $\neg\beta$ is equivalent in T to a disjunction wnfv of S-equations and S-relations,*
3. *for all $x \in V$*
   - *the S-formula $\exists x\, \beta$ is equivalent in T to false, or to a wnfv conjunction of S-relations,*
   - *for all $x \in V$, we have $T \models \exists_{o\,\infty}^{\{faux\}} x\, \beta$.*

Let us present now the main result of this chapter:

Theorem 4.2.1.5 *If T is flexible then T^* is complete.*

To show this theorem we will first introduce structured formulas much more complex than the conjunctions of atomic formulas and that we call *blocks*. We will then show using these blocks that if T is flexible then T^* is zero-infinite-decomposable and thus complete.

4.2.2 Blocks and solved blocks in T^*

Definition 4.2.2.1 *A* block *is a conjunction α of S^*-formulas of the form*

- *true, false, px, $\neg px$,*
- *$x = y$, $x = fx_1 \ldots x_n$, with $f \in F^*$,*
- *$t_1 = t_2 \wedge \bigwedge_{i=1}^{n} px_i$, where t_1 and t_2 are S-terms and $var(t_1 = t_2) = \{x_1, \ldots, x_n\}$,*
- *$rt_1 \ldots t_n$, with $r \in R$ and the t_i S-terms,*

and such that α contains px or $\neg px$ for all variables $x \in$ var(α). A relational block *is a block which does not contain S^*-equations. An* equational block *is a block which does not contain S-relations and where each variable has at least an occurrence in an S^*-equation.*

Example 4.2.2.2 *Let us consider the S^*-theory ${T_{ord}}^*$. The following S^*-formula is a block*

$$x = fxy \wedge z = gxy \wedge \neg px \wedge py \wedge pz.$$

While the S^*-formula

$$fxy = gyx \wedge px \wedge py,$$

is not a block because fxy and gyx are not S-terms but S^*-terms. The S^*-formula

$$fxy < gyx \wedge px \wedge py,$$

is not a block because fxy and gyx are S^*-terms and not S-terms. The block

$$x = fxy \wedge y = z \wedge \neg px \wedge py \wedge pz,$$

is an equational block, while the block

$$x = fxy \wedge y = z \wedge \neg px \wedge py \wedge pz \wedge pw,$$

is not equational because the variable w does not occur in any equation of this block.

Definition 4.2.2.3 *Let α be a block and $\bar{x}$ be a vector of variables. A variable u is called* reachable *in $\exists\bar{x}\alpha$ if u is a free variable in $\exists\bar{x}\alpha$, or α has a sub-formula of the form $y = t(u) \wedge \neg p\, y$ with $t(u)$ an S^*-term containing u and y a reachable variable. In the last case, the equation $y = t(u)$ is called reachable in $\exists\bar{x}\alpha$.*

Example 4.2.2.4 *Let us consider the S^*-theory ${T_{ord}}^*$. In the formula*

$$\exists yz\, x = fxy \wedge y = z \wedge \neg px \wedge py \wedge pz,$$

the variables x and y as well as the equation $x = fxy$ are reachable. The variable z as well as the equation $y = x$ are not reachable because the preceding formula does not contain sub-formulas of the form $\neg py$.

From the general axiomatization of T^*, given in Section 4.1.1, and more exactly from axioms 1 and 2, we have the following property

Property 4.2.2.5 *Let α be a block. If all the variables of $\bar{x}$ are reachable in $\exists\bar{x}\alpha$, then $T^* \models \exists?\bar{x}\alpha$.*

Definition 4.2.2.6 A block α is called *well-typed* if α does not contain sub-formulas of the form:

- $p\,x \wedge \neg p\,x$,
- $x = h\bar{y} \wedge p\,x$, with $h \in F^* - F$,
- $x = f_0 \wedge \neg p\,x$, with f_0 a constante of F,
- $x_0 = fx_1 ... x_n \wedge \neg p\,x_0 \wedge \bigwedge_{i=1}^{n} p\,x_i$, with $f \in F$,
- $x_0 = fx_1 ... x_n \wedge p\,x_0 \wedge \neg p\,x_i$, with $f \in F^*$
- $x_0 = x_1 \wedge p\,x_0 \wedge \neg p\,x_1$,
- $x_0 = x_1 \wedge \neg p\,x_0 \wedge p\,x_1$,
- $rt_1 ... t_n \wedge \neg p\,x_i$ with $r \in R$ and $x_i \in var(rt_1 ... t_n)$.

Definition 4.2.2.7 *Let α be a well-typed block. An S^*-equation of α of the form $t_1 = t_2$ is called* tree-equation *in α if for all $x \in var(t_1 = t_2)$, px is a sub-formula of α. It is called* non-tree-equation *in α if there exists $x \in var(t_1 = t_2)$ such that $\neg px$ is a sub-formula of α.*

In a block well-typed α every equation is either a tree-equation or a non-tree-equation. This property holds since in a well-typed block there exists no sub-formulas of the form $\neg px \wedge px$. Note also that all the non-tree-equations of a well-typed block α are of the form $x = y$ or $x = f\bar{y}$ with $f \in F^*$.

Definition 4.2.2.8 *Let α be a well-typed block. Let $x = t$, with t a term, be an S^*-tree-equation of α. The variable x is called α-leader of the S^*-equation $x = t$. Let $t_1 = t_2$, with t_1 and t_2 two S-terms, be an S^*-non-tree-equation of α. We call α-leader of the S^*-equation $t_1 = t_2$ the greatest variable x_k in $var(t_1 = t_2)$ according to the order $\succ$ such that $T \models \exists! x_k \, t_1 = t_2$.*

Example 4.2.2.9 *Let us consider the theories T_{ord} and $T_{ord}{}^*$. Let x, y, z be variables with $x \succ y \succ z$. Let α be the block*

$$x = fxy \wedge z = y \wedge \neg px \wedge py \wedge pz.$$

The variable x is α-leader of the S^*-equation $x = fxy$. The variable y is α-leader of the S^*-equation $z = y$ because $T_{ord} \models \exists! y \, z = y$ and $y \succ z$.

Definition 4.2.2.10 *A block α is called* ($\succ$)-solved *in T^* if*

1. *α is a well-typed block which does not contain sub-formulas of the form false $\wedge \beta$ with β a formula different from the formula true,*
2. *each S^*-equation of α has a distinct α-leader which does not occur in the S-relations of α,*
3. *every conjunction of S-equations and S-relations is formated in T.*

Note that from the last point of this definition and according to the definition of the formated formulas in T, we deduce that if $x = y$ is a sub-formula of the ($\succ$)-solved block α then $x \succ y$. Note also that every S^*-equation of the form $x = y$ is also an S-equation.

Example 4.2.2.11 *Let us consider the theory $T_{ord}{}^*$. Let x, y, z be variables with $x \succ y \succ w \succ z$. The block*

$$x = fxy \wedge y = w \wedge w = z \neg px \wedge py \wedge pz \wedge pw,$$

is not ($\succ$)-solved because w is the leader of the S-equation $w = z$ and occurs also in the S-equation $y = w$. The blocks false, true, and

$$x = fxy \wedge y = z \wedge w = z \wedge \neg px \wedge py \wedge pz \wedge pw,$$

are ($\succ$)-solved.

Property 4.2.2.12 *If T is flexible then every block is equivalent in T^* to a wnfv ($\succ$)-solved block.*

Proof. Let us introduce the following rewriting rules which transform a block into a wnfv ($\succ$)-solved block in T^* for every flexible theory T. To apply the rule $p_1 \Longrightarrow p_2$ to the block

p means to replace in p, a sub-formula p_1 by the formula p_2, by considering the connector $\wedge$ associative and commutative.

$$
\begin{array}{lll}
(1) & p\,x \wedge \neg p\,x & \Longrightarrow \; \mathit{false} \\
(2) & x = h\bar{y} \wedge p\,x & \Longrightarrow \; \mathit{false} \\
(3) & x = f_0 \wedge \neg p\,x & \Longrightarrow \; \mathit{false} \\
(4) & x_0 = f x_1 ... x_n \wedge \neg p\,x_0 \wedge \bigwedge_{i=1}^{n} p\,x_i & \Longrightarrow \; \mathit{false} \\
(5) & x_0 = g x_1 ... x_n \wedge p\,x_0 \wedge \neg p\,x_i & \Longrightarrow \; \mathit{false} \\
(6) & x_0 = x_1 \wedge p\,x_0 \wedge \neg p\,x_1 & \Longrightarrow \; \mathit{false} \\
(7) & x_0 = x_1 \wedge \neg p\,x_0 \wedge p\,x_1 & \Longrightarrow \; \mathit{false} \\
(8) & r t_1 ... t_n \wedge \neg p\,z & \Longrightarrow \; \mathit{false} \\
(9) & \mathit{false} \wedge \alpha & \Longrightarrow \; \mathit{false}, \\
(10) & x = f_1 y_1 ... y_m \wedge x = f_2 z_1 ... z_n \wedge \neg p x & \Longrightarrow \; \mathit{false}, \\
(11) & x = x & \Longrightarrow \; \mathit{true} \\
(12) & x = g y_1 ... y_n \wedge x = g z_1 ... z_n \wedge \neg p x & \Longrightarrow \; x = g y_1 ... y_n \wedge \bigwedge_{i \in 1..n} y_i = z_i \wedge \neg p x, \\
(13) & x = y \wedge x = g z_1 ... z_n \wedge \neg p x & \Longrightarrow \; x = y \wedge y = g z_1 ... z_n \wedge \neg p x, \\
(14) & x = y \wedge x = z \wedge \neg p x & \Longrightarrow \; x = y \wedge z = y \wedge \neg p x \\
(15) & y = x \wedge \neg p x & \Longrightarrow \; x = y \wedge \neg p x \\
(16) & \alpha \wedge \bigwedge_{i \in I} p x_i & \Longrightarrow \; \alpha' \wedge \bigwedge_{i \in I} p x_i
\end{array}
$$

with $h \in F^* - F$, f_0 a constant of F, $f \in F$, $g \in F^*$ and f_1 and f_2 two distinct elements of F^*. In the rule (8), $r \in R$ and $z \in var(rt_1...t_n)$, the rules (13), (14) and (15) are applied only if $x \succ y$. In the rule (16), α is a non-formated conjunction in T of S-atomic formulas, $var(\alpha) = \{x_1..., x_n\}$, $I = \{1, ..., n\}$ is a finite possibly empty set and α' is a formated conjunction (according to the order $\succ$) of S-atomic formulas equivalent to α in T[20]. Let us show now that every repeated application of the preceding rules on a block α terminates, keeps the equivalence in T^* and produces a wnfv ($\succ$)-solved block β.

Proof first part: every repeated application of the rules on a block terminates. Since the variables which occur in our formulas are ordered by the linear dense order relation $\succ$, we can number them by positive integers such that $x \succ y \leftrightarrow no(x) > no(y)$, where $no(x)$ is the positive integer associated to the variable x. Let us consider the 5-tuple $(n_1, n_2, n_3, n_4, n_5)$ where the n_i are the following non-negative integers:

- n_1 is the number of sub-formulas of the form $x = fy_1...y_n$, with $f \in F^*$,
- n_2 is a function which gives 1 if the formula contains a non-formated conjunction in T of S-atomic formulas and 0 otherwise,
- n_3 is the number of occurrences of atomic formulas,
- n_4 is the sum of $no(x)$ for every occurrence of a variable x,
- n_5 is the number of sub-formulas of the form $y = x$ with $x \succ y$.

for each rule there exists a row i such that the application of this rule decreases or does not change the value of the n_j with $1 \leq j < i$, and decreases the value of n_i. The row i is equal to: 3 for the rules (1)...(10), 4 for the rule (11), 1 for the rule (12), 4 for the rules (13) and (14), 5 for the rule (15) and 2 for the rule (16). To each sequence of formulas obtained by finite application of the rules, we can associate a series of 5-tuples $(n_1, n_2, n_3, n_4, n_5)$ which is strictly decreasing

[20] The formula α' always exists since T is flexible.

in the lexicographic order. Since these n_i's are positive integers they can not be negative and thus this series is finite and the application of our rules terminates.
Proof, second part: The rules keep the equivalence in T^*. The rule (1) is evident in T^*. The rule (2) comes from axiom 6 of T^*. The rules (3) and (4) come from axiom 5 of T^*. The rule (5) comes from axioms 5 and 6. The rules (6) and (7) come from the properties of the equality. The rule (8) comes from axiom 4 of T^*. The rule (9) is evident. The rule (10) comes from axiom 2 of T^*. The rule (11) is evident in T^*. The rule (12) comes from axiom 1 of T^*. The rules (13), (14) and (15) are evident in T^* and come from the properties of the equality. The rule (16) holds since T is flexible and using axioms 4,5 and 8 of T^* which enable to move from properties on T to properties on T^* by introducing typing constraints.
Proof third part: every finite application of these rules on a block produces a ($\succ$)-solved block equivalent in T^*. Let us suppose that the obtained formula is not a ($\succ$)-solved block and no-rules can be applied. Thus, at least one of the three conditions of Definition 4.2.2.10 does not hold. According to which condition 1 or 2 or 3 does not hold one at least of the rules (1),..., (9) or (10),(12),(13),(14), (16) or (11),(15), (16) can be applied which contradicts our supposition.
□

Property 4.2.2.13 *Let α be an equational ($\succ$)-solved block, different from the formula false. Let α^* be the conjunction of the sub-formulas of α of the form py or $\neg py$ with y a variable of α which is not α-leader in the S^*-equations of α. Let $\bar{x}$ be the set of the α-leaders of the S^*-equations of α. We have $T^* \models \alpha^* \rightarrow \exists! \bar{x}\alpha$.*

This property comes from axiom 3 of T^* and using the fact that the block α is ($\succ$)-solved.

Example 4.2.2.14 *Let us consider the theory $\mathcal{T}_{ord}$. We have*

$$T^*_{ad} \models pw \wedge pz \rightarrow (\exists! xy\, x = fxw \wedge y = z \wedge \neg px \wedge py \wedge pz \wedge pw).$$

But we have not

$$T^*_{ad} \models \exists! xy\, x = fxw \wedge y = z \wedge \neg px \wedge py \wedge pz \wedge pw$$

because if we instantiate z by a tree-value for example $f1$ (f a 1-ary function symbol taken from $F^* - \{+, -, 0, 1\}$) then y will be a tree which contradicts the fact that we have py.

4.2.3 T^* is zero-infinite-decomposable

To show Theorem 4.2.1.5, it is enough to show the following theorem:

Theorem 4.2.3.1 *If T is flexible then T^* is zero-infinite-decomposable.*

Proof. Let T be a flexible theory. Let us show that T^* satisfies the fifth conditions of Definition 3.2.1.1. Let us denote by F_0 the set of the constants of F. The sets $\Psi(u)$, A, A' and A'' are chosen as follows:

Choice of the sets $\Psi(u)$, A, A' and A''

- $\Psi(u)$ is the set of the S^*-formulas of the form $\exists \bar{y}\, u = f\bar{y} \wedge \neg p\, u$, with f a function symbol taken from $F^* - F_0$.
- A is the set of the blocks.
- A' is the set of the S^*-formulas of the form $\exists \bar{x}'\alpha'$, where

- α' is an equational ($\succ$)-solved block, different from the formula *false*, and such that the order $\succ$ is such that all the variables of $\bar{x}'$ are greater than the free variables of $\exists \bar{x}' \alpha'$,
- all the variables of $\bar{x}'$ and all the S^*-tree-equations of α' are reachable in $\exists \bar{x}' \alpha'$,
- all the variables of the S^*-non-tree-equations of α' are reachable in $\exists \bar{x}' \alpha'$,

- A'' is the set of the ($\succ$)-solved relational blocks.

Note that the set A is closed for the conjunction and A'' is a sub-set of A.

T^* satisfies the first condition of Definition 3.2.1.1

Let us show that every formula of the form $\exists \bar{x}\, \alpha \wedge \psi$, with $\alpha \in A$ and ψ any formula is equivalent in T^* to a wnfv S^*-formula of the form

$$\exists \bar{x}'\, \alpha' \wedge (\exists \bar{x}''\, \alpha'' \wedge (\exists \bar{x}'''\, \alpha''' \wedge \psi))), \tag{4.1}$$

with $\exists \bar{x}' \alpha' \in A'$, $\alpha'' \in A''$, $\alpha''' \in A$ and $T^* \models \forall \bar{x}''\, \alpha'' \rightarrow \exists! \bar{x}''' \alpha'''$.

Let us choose the order $\succ$ such that all the variables of $\bar{x}$ are greater than the free variables of $\exists \bar{x} \alpha$. Let β be the ($\succ$)-solved block of α, (β exists according to Property 4.2.2.12). Let X be the set of the variables of the vector $\bar{x}$. Let Y_{rea} be the set of the reachable variables in $\exists \bar{x} \beta$ and let Y_{nrea} be the set of the non reachable variables in $\exists \bar{x} \beta$. Let us rename the variables of $Y_{nrea} \cap X$ which have at least one occurrence in a non-tree-equation of β by variables greater than all the other variables of β. Note that these variables are quantified and thus we can rename them and keep the equivalence. Let β^* be the ($\succ$)-solved block of β. Let *Lead* be the set of the β^*-leaders of the S^*-equations of β^*. If $faux$ is a sub-formula of β^* then $\bar{x}' = \bar{x}'' = \bar{x}''' = \varepsilon$, $\alpha' = true$, $\alpha'' = false$ and $\alpha''' = true$. Else
– $\bar{x}'$ contains the variables of $X \cap Y_{rea}$,
– $\bar{x}''$ contains the variables of $(X - Y_{rea}) - Lead$.
– $\bar{x}'''$ contains the variables of $(X - Y_{rea}) \cap Lead$.
– α' is of the form $\alpha'_1 \wedge \alpha'_2$ where α'_1 is the conjunction of (1) all the tree-equations of β^* which are reachable in $\exists \bar{x} \beta^*$, (2) all the non-tree-equations of β^* whose β^*-leader is not element of $Y_{nacc} \cap X$. The formula α'_2 is the conjunction of all the sub-formulas of β^* of the form $p\,x$ or $\neg p\,x$ with x having at least an occurrence in α'_1.
– α'' is of the form $\alpha''_1 \wedge \alpha''_2$ where α''_1 is the conjunction of all the sub-formulas of β^* of the form $p\,x$ or $\neg p\,x$ with $x \notin \bar{x}'''$. The formula α''_2 is the conjunction of all the sub-formulas of β^* of the form $r t_1 ... t_n$ with $r \in R$ and t_i S-terms.
– α''' is of the form $\alpha'''_1 \wedge \alpha'''_2$ where α'''_1 is the conjunction of (1) all the S^*-tree-equations of β^* which are not reachable in $\exists \bar{x} \beta^*$, (2) all the S^*-non-tree-equations of β^* whose β^*-leaders belong to $Y_{nrea} \cap X$. The formula α'''_2 is the conjunction of all the sub-formulas of β^* of the form $p\,x$ or $\neg p\,x$ with x having at least an occurrence in α'''_1.

According to our construction, it is clear that $\exists \bar{x}' \alpha' \in A'$, $\alpha'' \in A''$ and $\alpha''' \in A$. Moreover, according to axiom 3 of T^* and Property 4.2.2.13 we have $\mathcal{T}_{ord} \models \forall \bar{x}'' \alpha'' \rightarrow \exists! \bar{x}''' \alpha'''$. Let us show now that (4.1) and $\exists \bar{x} \alpha \wedge \psi$ are equivalents in T^*. Let X', X'' and X''' be the sets of the variables of the vectors[21] of $\bar{x}'$, $\bar{x}''$ and $\bar{x}'''$. If β^* is the formula *false*, then the equivalence of the decomposition is evident. Else, β^* is a ($\succ$)-solved block which does not contain the sub-formula *false*. Thus, according to our construction we have $X = X' \cup X'' \cup X'''$, $X' \cap X'' = \emptyset$,

[21] Of course, if $\bar{x} = \varepsilon$ then $X = \emptyset$

$X' \cap X''' = \emptyset$, $X'' \cap X''' = \emptyset$, for all $x_i'' \in X''$ we have $x_i'' \notin var(\alpha')$ and for all $x_i''' \in X'''$ we have $x_i''' \notin var(\alpha' \wedge \alpha'')$. These properties come from the definition of $(\succ)$-solved block and the order $\succ$ which has been chosen such that the quantified non-reachable variables are greater than the quantified reachable variables which are greater than the free variables in $\exists \bar{x} \beta^*$. On the other hand, each S^*-equation and each S^*-relation of β^* occurs in $\alpha' \wedge \alpha'' \wedge \alpha'''$ and each S^*-equation and each S^*-relation of $\alpha' \wedge \alpha'' \wedge \alpha'''$ occurs in β^* and thus $T^* \models \beta^* \leftrightarrow (\alpha' \wedge \alpha'' \wedge \alpha''')$. We have shown that the quantifications are coherent and the equivalence $T^* \models \beta^* \leftrightarrow \alpha' \wedge \alpha'' \wedge \alpha'''$ holds. According to Property 4.2.2.12 we have $T^* \models \alpha \leftrightarrow \beta^*$ and thus the decomposition keeps the equivalence in T^*.

T^* satisfies the second condition of Definition 3.2.1.1

Let us show that T^* satisfies the second condition of Definition 3.2.1.1, i.e. if $\exists \bar{x}' \alpha' \in A'$ then $T^* \models \exists ? \bar{x}' \alpha'$. Since $\exists \bar{x}' \alpha' \in A'$ and according to the choice of the set A', the variables of $\bar{x}'$ are reachable in $\exists \bar{x}' \alpha'$. Thus, according to Property 4.2.2.5 we get $T^* \models \exists ? \bar{x}' \alpha'$.

Let us show now that if y is a free variable in $\exists \bar{x}' \alpha'$ then $T^* \models \exists ? y \bar{x}' \alpha'$, or there exists $\psi(u) \in \Psi(u)$ such that $T^* \models \forall y\, (\exists \bar{x}'\, \alpha') \rightarrow \psi(y)$. Let y be a free variable of $\exists \bar{x}' \alpha'$. Since α' is an equational $(\succ)$-solved block different from *false*, then three cases arise:

Either, y occurs in a sub-formula of α' of the form $y = t(\bar{x}', \bar{z}', y) \wedge \neg py$, where $\bar{z}'$ is the set of the free variables of $\exists \bar{x}' \alpha'$ which are different from y, $t(\bar{x}', \bar{z}', y)$ is a term which begins by an element of $F^* - F_0$, followed by variables taken from $\bar{x}'$ or $\bar{z}'$ or $\{y\}$. In this case, the formula $\exists \bar{x}' \alpha'$ implies in T^* the formula

$$\exists \bar{x}'\, y = t(\bar{x}', \bar{z}', y) \wedge \neg p\, y,$$

which implies in T^* the formula

$$\exists \bar{x}' \bar{z}' w\, y = t(\bar{x}', \bar{z}', w) \wedge \neg p\, y, \tag{4.2}$$

where $y = t(\bar{x}', \bar{z}', w)$ is the formula $y = t(\bar{x}', \bar{z}', y)$ in which we have replaced every free occurrence of y in the term $t(\bar{x}', \bar{z}', y)$ by the variable w. According to the choice of the set $\Psi(u)$ defined in Section 4.2.3, the formula (4.2) belongs to $\Psi(y)$.

Or, y occurs in a sub-formula of α' of the form $y = z \wedge \neg py$. In this case, since y is α'-leader of the equation $y = z$, then we have $y \succ z$ (because α' is $(\succ)$-solved), and thus, z is a free variable in $\exists \bar{x}' \alpha'$ because the order $\succ$ is such that all the variables of $\bar{x}'$ are greater than the free variables of $\exists \bar{x}' \alpha'$ (thus greater than y). On the other hand, since α' is a $(\succ)$-solved block, y is not α'-leader in another equation of α' (because all the α'-leaders are distinct), thus the variable y can not occur in another left hand sides of an S^*-equation of α' (because $\neg py$ is a sub-formula of the well-typed block α'). Thus, since the variables of $\bar{x}$ are reachable in $\exists \bar{x}' \alpha'$ (according to the choice of the set A' in Section 4.2.3) then all the variables of $\bar{x}'$ remain reachable in $\exists \bar{x}' y\, \alpha'$. Moreover, for each value of the free variable z, there exists at most a value for y. Thus, according to Property 4.2.2.5 we have $T^* \models \exists ? \bar{x}' y\, \alpha'$.

Or, y occurs only in sub-formulas of the form

$$x_0 = t(y) \;\; or \;\; t_1 = t_2, \tag{4.3}$$

with $x_0 = t(y)$ an S^*-tree-equation of α', $t(y)$ an S^*-term which begins by an element of $f \in F^*$ and contains at least an occurrence of the variable y, and $t_1 = t_2$ an S^*-non-tree-equation of α' containing at least an occurrence of y. Let us recall that according to the choice of the set A' (section 4.2.3), $\bar{x}'$ contains the quantified reachable variables in $\exists \bar{x}'\, \alpha'$ and all the tree-equations of α' are reachable in $\exists \bar{x}'\alpha'$. Two cases arise: (1) If y occurs in a tree-equation of α', then since y does not occur in another left hand side of a tree-equation of α', then the variables of $\bar{x}'y$ remain reachable in $\exists \bar{x}'y\,\alpha'$ and thus according to Property 4.2.2.5 we get $T^* \models \exists ? \bar{x}'y\,\alpha'$. (2) If y occurs only in non-tree-equations of α', then according to the choice of the set A' the variables of $\bar{x}'$ are reachable in $\exists \bar{x}'\,\alpha'$. Since y does not occur in a tree-equation of α', then the variables of $\bar{x}'$ remain reachable in $\exists \bar{x}'y\,\alpha'$. Moreover, since α' is $(\succ)$-solved then its S-equations are formated and thus $T^* \models \exists ? y\,\alpha$ and thus according to Property 4.2.2.5 we get $T^* \models \exists ? \bar{x}'y\,\alpha'$.

In a ll the cases, T^* satisfies the second condition of Definition 2.2.1.1.

T^* satisfies the third condition of Definition 3.2.1.1

T^* satisfies the first point of the third condition of Definition 3.2.1.1

Let us show that if $\alpha'' \in A''$ then the formula $\neg\alpha''$ is equivalent in T^* to a disjunction of elements of A, i.e. to a disjunction of blocks. Let α'' be an S^*-formula which belongs to A''.

According to the choice of the set A'' given in Section 4.2.3, either α'' is the formula *false* and thus $\neg\alpha''$ is the formula *true* which belongs to A'', or α'' is a $(\succ)$-solved relational block of the form

$$\beta \wedge (\bigwedge_{x\in X} px) \wedge (\bigwedge_{y\in Y} \neg py),$$

with β a conjunction of S-relations of the form $rt_1...t_n$ with $r \in R$ and $var(\beta) \subseteq X$. According to the second point of the definition of flexible theory, we have $T \models \neg\beta \leftrightarrow \beta'$ where β' is a disjunction of S-relations and S-equations. Thus, according to the axiomatization of T^* (more exactly Axioms 4,5 and 8), the S^*-formula $\neg\alpha''$ is equivalent in T^* to an S^*-formula wnfv of the form

$$(\bigvee_{k\in K} (\beta'_k \wedge p_k)) \vee (\bigvee_{x\in X} \neg px) \vee (\bigvee_{y\in Y} py),$$

where each β'_k is either an S-equation or an S-relation, p_k is a conjunction of S^*-formulas of the form px for every variable $x \in var(\beta'_k)$. It is obvious that this formula is a disjunction of blocks. Thus T^* satisfies the first point of the third condition of Definition 3.2.1.1.

T^* satisfies the second point of the third condition of Definition 3.2.1.1

Let us show that if $\alpha'' \in A''$ then for every variable x'', the S^*-formula $\exists x''\alpha''$ is equivalent in T^* to an element of A''. Let α'' be an S^*-formula of A'', three cases arise:

(1) If x'' has no occurrences in α'', then the S^*-formula $\exists x''\,\alpha''$ is equivalent in T^* to α'' which belongs to A''.

(2) If the S^*-formula $\exists x''\,\alpha''$ is of the form $\exists x''\,\alpha''_1 \wedge \neg p\,x''$ with $\alpha''_1 \in A''$ and x'' has no occurrences in α''_1, then the S^*-formula $\exists x''\,\alpha''$ is equivalent in T^* to $\alpha''_1 \wedge (\exists x''\,\neg p\,x'')$, which according to axiom 3 of T^* is equivalent in T^* to α''_1, which belongs to A''.

(3) If the S^*-formula $\exists x''\,\alpha''$ is of the form

$$\exists x''\,\alpha''_1 \wedge \varphi,$$

with α''_1 a conjunction of S^*-relations with $x'' \notin var(\alpha''_1)$ and φ a relational block containing only S^*-relations of the form px'' or $rt_1...t_n$ with $r \in R$ and $x'' \in var(rt_1...t_n)$, then the formula $\exists x''\,\alpha''$ is equivalent in T^* to

$$\alpha''_1 \wedge \phi \wedge (\exists x''\,\varphi), \tag{4.4}$$

with ϕ the conjunction of the typing constraints of φ of the form px or $\neg px$ with $x \in var(\alpha_1'')$. Thus, the formula $\alpha_1'' \wedge \phi$ is a relational ($\succ$)-solved block. If φ is reduced to the formula px, then according to axiom 7 of T^*, the formula (4.4) is equivalent in T^* to $\alpha_1'' \wedge \phi$, which belongs to A''. Else, φ is of the form $\varphi_1 \wedge \varphi_2 \wedge px''$ where φ_1 is the conjunction of the typing constraints of φ which have no occurrences of x'', and φ_2 is the conjunction of the relations of φ of the form $rt_1...t_n$ with $r \in R$ and the t_i's S-terms. According to the last point of the definition of flexible theory, the formula $\exists x''\, \varphi_2$ is equivalent in T to *false* or to a conjunction φ_2' wnfv of S-relations. Thus, according to axioms 8 and 4 of T^*, the formula $\exists x''\, \varphi_1 \wedge \varphi_2 \wedge px''$ is equivalent in T^* either to *false* or to the relational ($\succ$)-solved block wnfv $\varphi_1 \wedge \varphi_2'$. Thus, the formula (4.4) is equivalent in T^* to *false* or to

$$\alpha_1'' \wedge \phi \wedge \varphi_1 \wedge \varphi_2',$$

which is a relational ($\succ$)-solved block. Thus, T^* satisfies the second point of the third condition of Definition 3.2.1.1.

T^* satisfies the third point of the third condition of Definition 3.2.1.1

Let us first introduce two properties which hold in each S^*-model M^* of T^*. The first one comes from the axiomatization of T^* and introduces the notion of *zero-infinite* in M^*. The second one comes from the last point of the definition of the flexible theories using also axioms 4 and 8 of T^*.

Property 4.2.3.2 *Let M^* be an S^*-model of T^* and $f \in F^* - F_0$. The set of the individuals i of M^*, such that $M^* \models \exists \bar{x}\, i = f\bar{x} \wedge \neg pi$, is infinite.*

Property 4.2.3.3 *Let M^* be an S^*-model of T^*. Let $\bigwedge_{j \in J} r_j(x)$ be a conjunction of S-relations, i.e. a conjunction of S-formulas of the form $rt_1...t_n$ with $r \in R$ and the t_i's S-terms. Let $\exists x \bigwedge_{j\in J} r_j'(x)$ be an instantiation of $\exists x \bigwedge_{j \in J} r_j(x)$ by individuals of M^*. Let $\varphi(x)$ be the formula*

$$p\,x \wedge \bigwedge_{j\in J} r_j'(x). \tag{4.5}$$

The set of the individuals i of M^* such that $M^* \models \varphi(i)$ is empty or infinite.

Let M^* be an S^*-model of T^*. Recall that $\Psi(u)$ is the set of the formulas of the form $\exists \bar{y}\, u = f\bar{y} \wedge \neg p\,u$, with $f \in F^* - F_0$. Let $\varphi(x)$ be a formula which belongs to A''. Let us show that for every variable x we have $T^* \models \exists_{o\,\infty}^{\Psi(u)} x\, \varphi(x)$. Let $\exists x\, \varphi'(x)$ be an instantiation of $\exists \bar{x}\, \varphi(x)$ by individuals of M^* such that $M^* \models \exists x\, \varphi'(x)$. Let us show that there exists an infinity of individuals i of M^* which satisfy

$$M^* \models \varphi'(i) \wedge \neg\psi_1(i) \wedge \cdots \wedge \neg\psi_n(i),$$

with $\psi_j(u) \in \Psi(u)$. This condition can be replaced by the following stronger one

$$M^* \models \begin{pmatrix} p\,i \vee \\ \psi_{n+1}(i) \end{pmatrix} \wedge \varphi'(i) \wedge \neg\psi_1(i) \wedge \cdot\cdot \wedge \neg\psi_n(i),$$

where $\psi_{n+1}(u)$ belongs to $\Psi(u)$ and has been chosen different from all the $\psi_1(u), \ldots, \psi_n(u)$, (always possible because $F^* - F$ is infinite according to the definition of F^*). Since for every k between 1 and n, we have

- $T^* \models p\,x \rightarrow \neg\psi_k(x)$
- $T^* \models \psi_{n+1}(x) \rightarrow \neg\psi_k(x)$ (axiom 2 of T^* conflict of symbols).

The preceding condition is simplified to

$$M^* \models (p\,i \wedge \varphi'(i)) \vee (\psi_{n+1}(i) \wedge \varphi'(i)).$$

Thus, knowing $M^* \models \exists x\, \varphi'(x)$, it is enough to show that there exists an infinity of individuals i of M^* such that

$$M^* \models p\,i \wedge \varphi'(i) \;\text{ or }\; M^* \models \psi_{n+1}(i) \wedge \varphi'(i). \tag{4.6}$$

two cases arise:

Either, the formula $p\,x$ occurs in $\varphi'(x)$. Since $\varphi'(x)$ is an instantiation of an equational $(\succ)$-solved block and $M^* \models \exists x\, \varphi'(x)$, then according to axiom 4 of T^*, we deduce that the S^*-formula $p\,x \wedge \varphi'(x)$ is equivalent in M^* to an S^*-formula of the form (4.5). According to Property 4.2.3.3 and since $M^* \models \exists x\, p\,x \wedge \varphi'(x)$, there exists an infinity of individuals i of M^* such that $M^* \models p\,i \wedge \varphi'(i)$ and thus such that (4.6).

Or, the S^*-formula $p\,x$ does not occur in $\varphi'(x)$. Since $\varphi'(x)$ is an instantiation of a relational $(\succ)$-solved block and $M^* \models \exists x\, \varphi'(x)$, then the S^*-formula $\psi_{n+1}(x) \wedge \varphi'(x)$ is equivalent in M^* to $\psi_{n+1}(x)$. According to Property 4.2.3.2, there exists an infinity of individuals i of M^* such that $M^* \models \psi_{n+1}(i)$, thus such that $M^* \models \psi_{n+1}(i) \wedge \varphi'(i)$ and thus such that (4.6).

In all the cases T^* satisfies the third condition of Definition 3.2.1.1.

T^* satisfies the fourth condition of Definition 3.2.1.1

Let us show that every conjunction of flat formulas is equivalent in T^* to a disjunction of elements of A. For that, it is enough to show that every flat formula is equivalent in T^* to a disjunction of blocks. Let φ be a flat formula. If it is of the form *true*, *false* or $p\,x$ then φ is a block. Else the following equivalences after distribution of the $\wedge$ on the $\vee$ give the needed combinations

$$T^* \models r x_0 ... x_n \leftrightarrow \begin{bmatrix} r x_0 ... x_n \wedge \\ \bigwedge_{i=0}^{n} (p\,x_i \vee \neg p\,x_i) \end{bmatrix},$$

$$T^* \models x_0 = x_1 \leftrightarrow \begin{bmatrix} x_0 = x_1 \wedge \\ \bigwedge_{i=0}^{1} (p\,x_i \vee \neg p\,x_i) \end{bmatrix},$$

$$T^* \models x_0 = f x_1 ... x_n \leftrightarrow \begin{bmatrix} x_0 = f x_1 ... x_n \wedge \\ \bigwedge_{i=0}^{n} (p\,x_i \vee \neg p\,x_i) \end{bmatrix},$$

with $r \in R$ and $f \in F^*$. Thus T^* satisfies the fourth condition of Definition 3.2.1.1.

T^* satisfies the fifth condition

Let us show that for every S^*-proposition φ of the form $\exists \bar{x}' \alpha' \wedge \alpha''$ with $\exists \bar{x}' \alpha' \in A'$ and $\alpha'' \in A''$, we have $\bar{x} = \varepsilon$, $\alpha' \in \{true, false\}$ and $\alpha'' \in \{true, false\}$. Since φ does not contain free variables, then there exists no reachable variables and no reachable equations in $\exists \bar{x}' \alpha'$. Thus, according to Section 4.2.3, we have $\bar{x}' = \varepsilon$. According to the choice of the set A', the S^*-formula α' is a $(\succ)$-solved block different from the formula *false*, thus since $\exists \varepsilon\, \alpha'$ does not contain free variables, then α' is the formula *true*[22]. Thus, the S^*-proposition φ is of the form $\exists\varepsilon\, true \wedge \alpha''$. According to the choice of the set A'' given in Section 4.2.3, α'' is a relational $(\succ)$-solved block. Since it does not contain free variables then it is either the formula *true*, or the formula *false*[23]. Thus, the theory T^* satisfies the fifth condition of Definition 3.2.1.1.

The theory T^* satisfies all the conditions of Definition 3.2.1.1. Thus it is zero-infinite-decomposable and thus complete. The theorem 4.2.1.5 is then proved. □

4.3 Extension into trees T^*_{ad} of ordered additive rational numbers

4.3.1 Axiomatization

Let $F = \{+, -, 0, 1\}$ be a set of function symbols of respective arities $2, 1, 0, 0$. Let $R = \{<\}$ a set of relation symbols containing only the binary relation symbol $<$. Let S be the signature $F \cup R$.

Notation 4.3.1.1 *Let a be a positive integer and the $t_1, ..., t_n$'s S-terms. Let us denote by*

- *Z the set of the integers,*
- *$t_1 < t_2$, the S-term $< t_1 t_2$,*
- *$t_1 + t_2$, the S-term $+t_1 t_2$,*
- *$t_1 + t_2 + t_3$, the S-term $+t_1(+t_2 t_3)$,*
- *$0.t_1$, the S-term 0,*
- *$-a.t_1$, the S-term $\underbrace{(-t_1) + \cdots + (-t_1)}_{a}$,*
- *$a.t_1$, the S-term $\underbrace{t_1 + \cdots + t_1}_{a}$.*

Let T_{ad} be the S-theory of ordered additive rational numbers. The axiomatization of T_{ad} consists in the set of the following S-propositions

1 $\forall x \forall y\, x + y = y + x$,
2 $\forall x \forall y \forall z\, x + (y + z) = (x + y) + z$,
3 $\forall x\, x + 0 = x$,
4 $\forall x\, x + (-x) = 0$,
5_n $\forall x\, n.x = 0 \rightarrow x = 0, \quad (n \neq 0)$
6_n $\forall x\, \exists! y\, n.y = x, \quad (n \neq 0)$
7 $\forall x\, \neg x < x$,
8 $\forall x \forall y \forall z\, (x < y \wedge y < z) \rightarrow x < z$,
9 $\forall x \forall y\, (x < y \vee x = y \vee y < x)$,
10 $\forall x \forall y\, x < y \rightarrow (\exists z\, x < z \wedge z < y)$,
11 $\forall x\, \exists y\, x < y$,
12 $\forall x\, \exists y\, y < x$,
13 $\forall x\, \forall y\, \forall z\, x < y \rightarrow (x + z < y + z)$,
14 $0 < 1$.

[22] The formula α' does not contain sub-formulas of the form $f_1 = f_2$ with f_1 and f_2 constants of F because α' is $(\succ)$-solved and thus all the S-equations are formated.

[23] The formula α'' does not contain sub-formulas of the form $r f_1 ... f_n$ with $r \in R$ and f_i constants of F because α' is $(\succ)$-solved and thus all the S-relations are formated.

with n a non-nul integer.

Property 4.3.1.2

$$T_{ad} \models \sum_{i=1}^{n} a_i.x_i = a_0.1 \leftrightarrow a_k.x_k = \sum_{i=1, i\neq k}^{n} (-a_i).x_i + a_0.1,$$

for every $k \in \{1, ..., n\}$.

Let F^* be an infinite set of function symbols containing the set $\{+, -, 0, 1\}$. Let $R^* = \{<, p\}$ be a set of relation symbols containing the symbol $<$ as well as the relation symbol p. Let S^* be the signature $F^* \cup R^*$. According to the transformations of axioms given in Section 4.1.1, the axiomatization of T^*_{ad} is the infinite set of the following S^*-propositions:

1 $\forall\bar{x}\forall\bar{y}\, ((\neg p\, f\bar{x}) \wedge (\neg p\, f\bar{y}) \wedge f\bar{x} = f\bar{y}) \rightarrow \bigwedge_i x_i = y_i,$

2 $\forall\bar{x}\forall\bar{y}\, f\bar{x} = g\bar{y} \rightarrow p\, f\bar{x} \wedge p\, g\bar{y},$

3 $\forall\bar{x}\forall\bar{y}\, (\bigwedge_{i\in I} p\, x_i) \wedge (\bigwedge_{j\in J} \neg p\, y_j) \rightarrow (\exists ! \bar{z}\, \bigwedge_{k\in K} (\neg p\, z_k \wedge z_k = t_k(\bar{x}, \bar{y}, \bar{z}))),$

4 $\forall x \forall y\, x < y \rightarrow (p\, x \wedge p\, y),$

5 $\forall x \forall y\, p\, x + y \leftrightarrow p\, x \wedge p\, y,$

6 $\forall x\, p\, -x \leftrightarrow p\, x,$

7 $\forall\bar{x}\, \neg p\, h\bar{x},$

8 $\forall x \forall y\, (p\, x \wedge p\, y) \rightarrow x + y = y + x,$

9 $\forall x \forall y \forall z\, (p\, x \wedge p\, y \wedge p\, z) \rightarrow x + (y + z) = (x + y) + z,$

10 $\forall x\, p\, x \rightarrow x + 0 = x,$

11 $\forall x\, p\, x \rightarrow x + (-x) = 0,$

12_n $\forall x\, p\, x \rightarrow (nx = 0 \rightarrow x = 0), \quad (n \neq 0)$

13_n $\forall x\, p\, x \rightarrow \exists ! y\, p\, y \wedge ny = x, \quad (n \neq 0)$

14 $\forall x\, p\, x \rightarrow \neg x < x,$

15 $\forall x \forall y \forall z\, p\, x \wedge p\, y \wedge p\, z \rightarrow ((x < y \wedge y < z) \rightarrow x < z),$

16 $\forall x \forall y\, (p\, x \wedge p\, y) \rightarrow (x < y \vee x = y \vee y < x),$

17 $\forall x \forall y\, (p\, x \wedge p\, y) \rightarrow (x < y \rightarrow (\exists z\, p\, z \wedge x < z \wedge z < y)),$

18 $\forall x\, p\, x \rightarrow (\exists y\, p\, y \wedge x < y),$

19 $\forall x\, p\, x \rightarrow (\exists y\, p\, y \wedge y < x),$

20 $\forall x\, \forall y\, \forall z\, (p\, x \wedge p\, y \wedge p\, z) \rightarrow (x < y \rightarrow (x + z < y + z)),$

21 $0 < 1,$

with f and g two distinct function symbols taken from F^*, $h \in F^* - F$, x, y, z variables, $\bar{x}$ a vector of variables x_i, $\bar{y}$ a vector of variables y_i, $\bar{z}$ a vector of distinct variables z_i and $t_k(\bar{x}, \bar{y}, \bar{z})$ an S^*-term which begins by a function symbol f_k element of F^* followed by variables taken from $\bar{x}$ or $\bar{y}$ or $\bar{z}$. Moreover, if $f_k \in F$ then $t_k(\bar{x}, \bar{y}, \bar{z})$ contains at least a variable of $\bar{y}$ or $\bar{z}$. A similar theory has been introduced by A. Colmerauer to model the execution of Prolog III and Prolog IV [6].

Note that the theory of trees and the theory of additive ordered rational numbers have non-disjoint signatures. In fact, the symbols $+$ and $-$ are tree constructors in the theory of trees and operations of addition and subtraction in the theory of additive ordered rational numbers. Note also that T^*_{ad} does not accept full elimination of quantifiers. For example, the S^*-formula $\exists x\, y = f x$ with $f \in F - \{+, -, 0, 1\}$ can not be simplified anymore in T^*_{ad}.

4.3.2 Completeness

Theorem 4.3.2.1 *The extension into trees T^*_{ad} of ordered additive rational numbers T_{ad} is a complete theory.*

According to Theorem 4.2.1.5, it is enough to show that T_{ad} is flexible to get the completeness of T^*_{ad}. Thus, let us show the following property

Property 4.3.2.2 *The theory T_{ad} of ordered additive rational numbers is a flexible theory.*

Proof. Let us show that T_{ad} satisfies the three conditions of Definition 4.2.1.4. In order to simplify this proof, we will remove the prefix S from the words: equations, relations, terms, formulas, since we will handle only the theory T_{ad} of signature S.

Let us denote by $\sum_{i=1}^n t_i$, the term $\overline{t_1 + t_2 + ... + t_n} + 0$, where $\overline{t_1 + t_2 + ... + t_n}$ is the term $t_1+t_2+...+t_n$ in which we have removed all the terms t_i which are equal to 0. For $n = 0$ the term $\sum_{i=1}^n t_i$ is reduced to the term 0. Formulas of the form $\sum_{i=1}^n a_i.x_i = a_0.1$ and $\sum_{i=1}^n a_i.x_i < a_0.1$ with $a_i \in Z$ are called *blocks* in T_{ad}. According to Definition 4.2.1.1, and since for all $x_j \in var(\sum_{i=1}^n a_i.x_i = a_0.1)$ with $a_j \neq 0$ we have $T_{ad} \models \exists! x_j \sum_{i=1}^n a_i.x_i = a_0.1$, (Property 4.3.1.2 and axiom 6_n of T_{ad}), then the leader of an equation of the form $\sum_{i=1}^n a_i.x_i = a_0.1$ is quit simply the greatest variable x_k with $k \in \{1, ..., n\}$ such that $a_k \neq 0$.

T_{ad} satisfies the first condition of Definition 4.2.1.4

Let us show that every conjunction of equations and relations is equivalent in T to a formated conjunction of atomic formulas wnfv, i.e. to a conjunction α wnfv of atomic formulas such that

1. α does not contain sub-formulas of the form $f_1 = f_2$ or $r f_1...f_n$ or $y = x$, where all the f_i belong to $\{0, 1\}$, $r \in \{<\}$ and $x \succ y$,
2. each equation of α has a distinct leader which have no occurrences in other equations or relations of α,
3. if α' is the conjunction of the equations of α then for all $x \in var(\alpha')$ we have $T_{ad} \models \exists ?x\, \alpha'$.

Let us introduce now the following rules that transform every conjunction of flat formulas either to *false*, or to a wnfv formated conjunction of blocks equivalent in T_{ad}.

$$
\begin{array}{lll}
(1) & \mathit{false} \wedge \alpha & \Longrightarrow \mathit{false} \\
(2) & 0 = 0.1 & \Longrightarrow \mathit{true} \\
(3) & 0 = a_0.1 & \Longrightarrow \mathit{false} \\
(4) & x = y & \Longrightarrow x + (-1).y = 0.1 \\
(5) & x = -y & \Longrightarrow x + y = 0.1 \\
(6) & x = y + z & \Longrightarrow x + (-1).y + (-1).z = 0.1 \\
(7) & \begin{bmatrix} \sum_{i=1}^n a_i.x_i = a_0.1 \wedge \\ \sum_{i=1}^n b_i.x_i = b_0.1 \end{bmatrix} & \Longrightarrow \begin{bmatrix} \sum_{i=1}^n a_i.x_i = a_0.1 \wedge \\ \sum_{i=1}^n (b_k a_i - a_k b_i).x_i = (b_k a_0 - a_k b_0).1 \end{bmatrix} \\
(8) & \begin{bmatrix} \sum_{i=1}^n a_i.x_i = a_0.1 \wedge \\ \sum_{i=1}^n b_i.x_i < b_0.1 \end{bmatrix} & \Longrightarrow \begin{bmatrix} \sum_{i=1}^n a_i.x_i = a_0.1 \wedge \\ \sum_{i=1}^n \lambda (b_k a_i - a_k b_i).x_i < (b_k a_0 - a_k b_0).1 \end{bmatrix}
\end{array}
$$

In the rule (3), $a_0 \neq 0$. In the rules (7) and (8), the variable x_k is the leader of the equation $\sum_i a_i.x_i = a_0.1$ and $b_k \neq 0$. In the rule (8), $\lambda = 1$ if $a_k > 0$ and $\lambda = -1$ otherwise. Of course, every repeated application of these rules terminates and produces either *false* or a formated conjunction wnfv of blocks equivalent in T_{ad}.

Let α be a conjunction of atomic formulas. By introducing quantified variables to transform the formulas into flat formulas, α is equivalent in T_{ad} to a formula of the form $\exists \bar{x}\, \beta$ with β a conjunction of flat formulas. Let us choose the order $\succ$ such that the variables of $\bar{x}$ are greater than free variables of $\exists\bar{x}\,\beta$. Let δ be the formula obtained from β after application of the preceding rules. Two cases arise:

Either, δ is the formula *false*, thus the formula $\exists\bar{x}\,\delta$ is equivalent to *false* in T_{ad}, thus the conjunction α is equivalent to *false* in T_{ad} which is a formated atomic formula.

Or, δ is a formated conjunction of blocks such that each variable of $\bar{x}$ has an occurrence as leader in an equation of δ. This restriction comes from the order $\succ$ which has been chosen such that the variables of $\bar{x}$ are greater than the free variables of $\exists\bar{x}\,\beta$. Thus, the formula δ is of the form

$$(\bigwedge_{i\in I}\delta_{x_i})\wedge\delta^*,$$

where each δ_{x_i} is an equation of δ whose leader x_i is a variable of $\bar{x}$ and where δ^* is a conjunction of blocks which does not contain occurrences of the variables of $\bar{x}$. The formula $\exists\bar{x}\beta$ is then equivalent in T_{ad} to

$$\delta^*\wedge(\exists\bar{x}\bigwedge_{i\in I}\delta_{x_i}),$$

which since each leader x_i does not occur in another equation, is equivalent in T_{ad} to

$$\delta^*\wedge\bigwedge_{i\in I}(\exists x_i\,\delta_{x_i}),$$

which since for each leader x_i we have $T_{ad}\models\exists!x_i\delta_{x_i}$, is equivalent in T_{ad} to δ^*. Thus, the formula α is equivalent to the formula δ^* which is a formated conjunction of blocks and thus a conjunction of atomic formulas. Then, the theory T_{ad} satisfies the first condition of Definition 4.2.1.4.

T_{ad} satisfies the second condition of Definition 4.2.1.4

Let us show that every formula of the form $\neg\alpha$ where α is a conjunction of relations is equivalent in T_{ad} to a disjunction of equations and relations. According to the preceding point, the formula α is equivalent in T_{ad} to a conjunction of blocks of the form $\sum_{j=1}^{n} b_j.x_j < b_0.1$. Since the order is linear then we have

$$T_{ad}\models\neg(\sum_{j=1}^{n} b_j.x_j < b_0.1)\leftrightarrow((\sum_{j=1}^{n}(-b_j).x_j<(-b_0).1)\vee(\sum_{j=1}^{n} b_j.x_j = b_0.1))$$

Thus, the formula $\neg\alpha$ is equivalent in T_{ad} to a disjunction of blocks, and thus to a disjunction of equations and relations. The theory T_{ad} satisfies the second condition of Definition 4.2.1.4.

T_{ad} satisfies the third condition of Definition 4.2.1.4

Let us show that for every conjunction of relations β and every variable x we have:

- the formula $\exists x\,\beta$ is equivalent in T_{ad} either to *false*, or to a wnfv conjunction of relations,
- $T_{ad}\models\exists_{o\,\infty}^{\{faux\}}x\,\beta$.

The first point is evident and comes from the Fourier elimination of quantifiers. The second point holds since the order is dense and without endpoints. Let M be a model of T_{ad}. For every instantiation $\exists x\,\beta'(i)$ of $\exists x\,\beta(i)$ by individuals of M, if $M\models\beta'(i)$ then there exists an infinity of individuals i of M such that $M\models\beta'(i)$, thus $T_{ad}\models\exists_{o\,\infty}^{\{faux\}}x\,\beta$.

The theory T_{ad} is flexible, and thus the extension into trees T^*_{ad} is zero-infinite-decomposable. Consequently it is complete according to Theorem 4.2.1.5. □

4.4 Discussion and partial conclusion

Since the theory of finite or infinite trees does not accept full elimination of quantifiers, then the extension into trees of any first-order theory T does not accept full elimination of quantifiers but has a big power of expressiveness and enables us to model hard problems in the form of first order constraints. We can cite for example the constraints representing k-winning positions in multi-players games introduced by A. Colmerauer and B. Dao [16, 7] which can be modeled easily using an extension into trees of additive integer numbers. The idea of the extension of the model of Prolog IV by other first order theories uses also extension into trees of first order theories. In fact, this work is a perspective of an extension of Prolog IV, by allowing the user to quantify the Prolog clauses and use our decision procedure to solve these clauses. Unfortunately, the two decision procedures given in Chapter 2 and 3 solve only propositions and can not present the solutions of the free variables in a clear and explicit way. A such work needs more complex definitions than those of the flexible theories as well as strong semantic conditions on the function and relation symbols of T. Let us then choose for example the theory T^*_{ad} of additive ordered rational numbers and let us try to give an efficient algorithm for solving any first order constraint in T^*_{ad} ! This will be our goal in Chapter 5.

Chapter 5

Solving first order constraints in T^*_{ad}

Contents

We present in this chapter an algorithm for solving general first order constraints in the theory T^*_{ad}. The algorithm is given in the form of 28 rewriting rules which transform every formula φ into a wnfv disjunction ϕ of solved formulas, equivalent to φ in T^*_{ad} and such that ϕ is, either the formula *true*, or the formula *false*, or a formula having at least one free variable and being equivalent neither to *true* nor to *false* in T^*_{ad}. While the decision procedures given in Chapter 2 and 3 solve only propositions and can not express clearly the solutions of the free variables, this algorithm can check formulas having free variables and being always true or false in T^*_{ad} and gives either *true* or *false*. Moreover, the solutions of the free variables of ϕ are always expressed in a clear and explicit way. We end this chapter by an example of solving a formula having free variables and being always false in T^*_{ad}. Note that the results presented in this chapter have been published in [14], [17], [21].

5.1 First order constraint in T^*_{ad}

5.1.1 A convenient axiomatization of T^*_{ad}

Let F be an infinite set of function symbols containing the symbols $\{+, -, 0, 1\}$ of respective arities $2, 1, 0, 0$. Let R be a set of relation symbols containing the 1-ary relation symbols *num* and *tree* .

Let a be a positive integer and $t_1, ..., t_n$ terms. Let us denote by:

- Z the set of the integers,
- $t_1 < t_2$, the term $< t_1 t_2$,
- $t_1 + t_2$, the term $+t_1 t_2$,
- $t_1 + t_2 + t_3$, the term $+t_1(+t_2 t_3)$,
- $0.t_1$, the term 0,
- $-a.t_1$, the term $\underbrace{(-t_1) + \cdots + (-t_1)}_{a}$,
- $a.t_1$, the term $\underbrace{t_1 + \cdots + t_1}_{a}$,
- $\Sigma_{i=1}^{n} t_i$, the term $\overline{t_1 + \ldots + t_n} + 0$ with $\overline{t_1 + \ldots + t_n}$ the term $t_1 + \ldots + t_n$ where all the terms equal to 0 have been removed. For $n = 0$ we get the term 0.

The theory T^*_{ad} of the extension into trees of ordered additive rational numbers consists in the set of the following propositions:

1 $\forall \bar{x} \forall \bar{y} \, (tree\, f\bar{x} \wedge tree\, f\bar{y} \wedge f\bar{x} = f\bar{y}) \rightarrow \bigwedge_i x_i = y_i$,
2 $\forall \bar{x} \forall \bar{y} \, f\bar{x} = g\bar{y} \rightarrow num\, f\bar{x} \wedge num\, g\bar{y}$,
3 $\forall \bar{x} \forall \bar{y} \, (\bigwedge_{i \in I} num\, x_i) \wedge (\bigwedge_{j \in J} tree\, y_j) \rightarrow (\exists! \bar{z} \bigwedge_{k \in K} (tree\, z_k \wedge z_k = t_k(\bar{x}, \bar{y}, \bar{z})))$,
4 $\forall x \forall y \, x < y \rightarrow (num\, x \wedge num\, y)$,
5 $\forall x \forall y \, num\, x + y \leftrightarrow num\, x \wedge num\, y$,
6 $\forall x \, num\, -x \leftrightarrow num\, x$,
7 $\forall \bar{x} \, tree\ h\bar{x}$,
8 $\forall x \forall y \, (num\, x \wedge num\, y) \rightarrow x + y = y + x$,
9 $\forall x \forall y \forall z \, (num\, x \wedge num\, y \wedge num\, z) \rightarrow x + (y + z) = (x + y) + z$,
10 $\forall x \, num\, x \rightarrow x + 0 = x$,
11 $\forall x \, num\, x \rightarrow x + (-x) = 0$,
12_n $\forall x \, num\, x \rightarrow (nx = 0 \rightarrow x = 0)$,
13_n $\forall x \, num\, x \rightarrow \exists! y \, num\, y \wedge ny = x$,
14 $\forall x \, num\, x \rightarrow \neg x < x$,
15 $\forall x \forall y \forall z \, num\, x \wedge num\, y \wedge num\, z \rightarrow ((x < y \wedge y < z) \rightarrow x < z)$,
16 $\forall x \forall y \, (num\, x \wedge num\, y) \rightarrow (x < y \vee x = y \vee y < x)$,
17 $\forall x \forall y \, (num\, x \wedge num\, y) \rightarrow (x < y \rightarrow (\exists z \, num\, z \wedge x < z \wedge z < y))$,
18 $\forall x \, num\, x \rightarrow (\exists y \, num\, y \wedge x < y)$,
19 $\forall x \, num\, x \rightarrow (\exists y \, num\, y \wedge y < x)$,
20 $\forall x \forall y \forall z \, (num\, x \wedge num\, y \wedge num\, z) \rightarrow (x < y \rightarrow (x + z < y + z))$,
21 $\forall x \, numx \leftrightarrow \neg tree\, x$
22 $0 < 1$,

where n is a non-nul integer, f and g are two distinct function symbols of F, $h \in F - \{+, -, 0, 1\}$, x, y, z are variables, $\bar{x}$ is a vector of variables x_i, $\bar{y}$ is a vector of variables y_i, $\bar{z}$ is a vector of distinct variables z_i and where $t_k(\bar{x}, \bar{y}, \bar{z})$ is a term which begins by a function symbol $f_k \in F - \{0, 1\}$ followed by variables taken from $\bar{x}$ or $\bar{y}$ or $\bar{z}$, moreover, if $f_k \in \{+, -\}$ then $t_k(\bar{x}, \bar{y}, \bar{z})$ contains at least a variable of $\bar{y}$ or $\bar{z}$.

This axiomatization is a more convenient version than those given in Chapter 4, in sense that it introduces explicitly the relation symbol *tree* to distinguish trees from rational numbers which will help us to present simple rules for solving first order constraints in T^*_{ad}.

From axioms 6,8,9,10,11 and 12 we have the following property

Property 5.1.1.1 *For every $k \in \{1, ..., n\}$ we have*

$$T \models (\Sigma_{i=1}^{k} a_i.x_i = a_0.1 \wedge \bigwedge_{i=1}^{k} num\, x_i) \leftrightarrow (a_k x_k = \Sigma_{i=1, i\neq k}^{k}(-a_i).x_i + a_0.1 \wedge \bigwedge_{i=1}^{k} num\, x_i).$$

5.1.2 Example of first order constraint in T^*_{ad}

Let us now introduce an example of *constraints* in T^*_{ad}. Let us consider the following two-player game: An ordered pair (n, m) of non-negative rational numbers is given and one after another each player subtracts 1 or 2 from n or m but keeping n and m non-negative. The first player who cannot play any more has lost.

Suppose that it is the turn of player A to play. A position (n, m) is called *k-winning* if, no matter the way the other player B plays, it is always possible for A to win, after having made at most k moves. The constraint defined in [7] expressing that a position x is k-winning is:

$$winning_k(x) \leftrightarrow \begin{bmatrix} \exists y move(x,y) \wedge \neg(\exists x move(y,x) \wedge \\ \neg(\exists y move(x,y) \wedge \neg(\exists x move(y,x) \wedge \neg(\ldots \wedge \\ \neg(\exists y move(x,y) \wedge \neg(\exists x move(y,x) \wedge \neg(false \underbrace{))\ldots)}_{2k} \end{bmatrix}$$

Each position (n, m) is represented by $c(i, j)$ with c a function symbol of arity 2 and $i, j \in \boldsymbol{Q}$. The constraint $move(x, y)$ is defined by

$$\begin{bmatrix} (\exists i \exists j\, x = c(i,j) \wedge y = c(i-1,j) \wedge i > 1 \wedge j > 0) \vee \\ (\exists i \exists j\, x = c(i,j) \wedge y = c(i-2,j) \wedge i > 2 \wedge j > 0) \vee \\ (\exists i \exists j\, x = c(i,j) \wedge y = c(i,j-1) \wedge i > 0 \wedge j > 1) \vee \\ (\exists i \exists j\, x = c(i,j) \wedge y = c(i,j-2) \wedge i > 0 \wedge j > 2) \vee \\ (\neg(\exists i \exists j\, x = c(i,j) \wedge num i \wedge num j) \wedge x = y) \end{bmatrix}$$

By replacing the definition of *move* in the constraint $winning_k(x)$, we have a first-order constraint with one free variable x in the theory T^*_{ad}. Solving this constraint means finding the positions x which are k-winning.

5.2 Blocks and quantified blocks in T^*_{ad}

5.2.1 Blocks and solved blocks in T^*_{ad}

Suppose that the variables of V are ordered by a linear dense order relation without endpoints, denoted by $\succ$. For every formula φ, the bounded variables are renamed such that in each sub-formulas of φ we have $x \succ y$ for each bounded variable x and each free variable y.

We call *leader* of the equation $x_0 = fx_1...x_n$ or $x_0 = x_1$, with $f \in F - \{0, 1\}$, the variable x_0. We call leader of the formula $\Sigma_{i=1}^{n} a_i.x_i = a_0.1$, with $a_i \in Z$ for all $i \in \{0, ..., n\}$, the greatest variable x_k in the order $\succ$ such that $a_k \neq 0$.

Let $f \in F$ and $a_i \in Z$ for all $i \in \{0, ..., n\}$. We call *basic formula* every conjunction α of formulas of the form

- *true*, *false*,
- $num\, x$, $tree\, x$,

- $x = y$, $x = fy_1 ... y_n$,
- $\Sigma_{i=1}^n a_i.x_i = a_0.1$, $\Sigma_{i=1}^n a_i.x_i < a_0.1$.

The formulas $num\, x$ and $tree\, x$ are called *typing constraints.*

Let α be a basic formula

(1) We say that $num\, x$ is *a consequence* of α if α contains at least a sub-formula of the following form

- $num\, x$, $x = y \wedge num\, y$, $x = 0$, $x = 1$,
- $y = x \wedge num\, y$, $x = -y \wedge num\, y$, $y = -x \wedge num\, y$,
- $z = y + x \wedge num\, z$, $z = x + y \wedge num\, z$, $x = y + z \wedge num\, z \wedge num\, y$,
- $\Sigma_{i=1}^n a_i.x_i = a_0.1$ or $\Sigma_{i=1}^n a_i.x_i < a_0.1$ with x one of the x_i.

(2) We say that $tree\, x$ is *a consequence* of α if α contains at least a sub-formula of the following form

- $x = y \wedge tree\, y$, $y = x \wedge tree\, y$,
- $x = -y \wedge tree\, y$, $y = -x \wedge tree\, y$,
- $x = y + z \wedge tree\, z$, $x = z + y \wedge tree\, z$, $y = x + z \wedge tree\, y \wedge num\, z$, $y = z + x \wedge tree\, y \wedge num\, z$,
- $tree\, x$, $x = hy_1 ... y_n$, with $h \in F - \{+, -, 0, 1\}$.

(3) We call *tree section* of α, the conjunction α_t of the sub-formulas of α of the form

- $true$, $tree\, x$,
- $x = y$ or $x = fy_1 ... y_n$, with $f \in F - \{0, 1\}$ and where x is such that $tree\, x$ is a sub-formula of α.

This section α_t is *formated* if the left hand sides of the equations of α_t are distinct and for each equation of the form $x = y$ of α_t we have $x \succ y$.

(4) We call *numeric section* of α, the conjunction α_n of the sub-formulas of α of the form

- $true$, $false$, $num\, x$,
- $\Sigma_{i=1}^n a_i.x_i = a_0.1$, $\Sigma_{i=1}^n a_i.x_i < a_0.1$,
- $x = y$, $x = -y$, $x = y + z$, where x is such that $num\, x$ is a sub-formula of α.

This section α_n is *consistent* if $T^*_{ad} \models \exists \bar{x}\, \alpha_n$ with $\bar{x} = var(\alpha_n)$. It is called *formated* if

- α_n does not contain sub-formulas of the form $x = y$, $x = -y$, $x = y + z$, $0 = a_0 1$, $0 < a_0 1$, with $a_0 \in Z$,
- α_n is consistent and each leader of an equation of α_n occurs in only one equation of α_n and does not occur in the inequations of α_n.

A variable u is called *reachable* in $\exists \bar{x} \alpha$ if u is a free variable in $\exists \bar{x} \alpha$, or α has a sub-formula of the form $y = t(u) \wedge tree\, y$ with $t(u)$ a term containing u and y a reachable variable. In the last case, the equation $y = t(u)$ is called reachable in $\exists \bar{x} \alpha$.

From axioms 1 and 2 of T^*_{ad} we have the following property

Property 5.2.1.1 *Let α be a basic formula. If all the variables of $\bar{x}$ are reachable in $\exists \bar{x}\, \alpha$ then $T^*_{ad} \models \exists ? \bar{x} \alpha$.*

We call *bloc* every basic formula α such that for each variable x in α, either $num\, x$ or $tree\, x$ is a sub-formula of α and α does not contain sub-formulas of the form

- $x = 0 \wedge tree\, x$, $x = 1 \wedge tree\, x$,
- $x = y \wedge num\, x \wedge tree\, y$, $x = y \wedge tree\, x \wedge num\, y$,
- $x = -y \wedge tree\, x \wedge num\, y$, $x = -y \wedge num\, x \wedge tree\, y$
- $x = y + z \wedge num\, x \wedge tree\, y$, $x = y + z \wedge num\, x \wedge tree\, z$, $x = h\bar{y} \wedge num\, x$,
- $x = y + z \wedge tree\, x \wedge num\, y \wedge num\, z$,
- $\Sigma_{i=1}^{n} a_i.x_i = a_0.1 \wedge tree\, x_k$, $\Sigma_{i=1}^{n} a_i.x_i < a_0.1 \wedge tree\, x_k$

with $h \in F - \{+, -, 0, 1\}$, $k \in \{1, ..., n\}$ and $a_i \in Z$ for all $i \in \{0, ..., n\}$.

Since each variable x in a block has a type of the form $num\, x$ or $tree\, x$, then every block α can be divided into two disjoint sections: a tree section and a numeric section.

A block α without equations is called *relational block*. A block α without inequations and where each variable has an occurrence in at least an equation of α is called *equational block*. A block α is *solved* if its tree sections and numeric section are formated.

Since in a solved block the numeric section is consistent and according to axioms 3 and 13_n, we have the following property

Property 5.2.1.2 *Let α be a solved block and $\bar{x}$ the vector of the variables of α. We have $T^*_{ad} \models \exists \bar{x} \alpha$.*

5.2.2 Properties of the solved blocks in T^*_{ad}

Property 5.2.2.1 *Let α be an equational solved block. Let $\bar{x}$ be the vector of the leaders of the equations of α. Let α^* be the conjunction of typing constraints of α of the form tree x or num x with x does not belong to $\bar{x}$. We have*

$$T \models \alpha^* \rightarrow \exists ! \bar{x}\, \alpha$$

Proof. This property is a consequence of axioms 3 and 13_n of T^*_{ad}. In fact, the equations of the tree section of α have distinct leaders, i.e. distinct left hand sides, and since the equations of the numeric section of α have distinct leaders which have an occurrence in one and only one equation of the numeric section of α, then by transforming all the equations of this numeric section into formulas of the form $a_k.x_k = \sum_{i=1, i \neq k} (-a_i).x_i + a_0.1$ where x_k is the leader of the equation $\sum_{i=1}^{n} a_i.x_i = a_0.1$ (Property 5.1.1.1), then the left hand sides of these equations are distinct and do not occur in other equations of the numeric section of α. Thus, in every model of T^*_{ad} and for every instantiation of the variables which occur in the right hand sides by individuals which respect the typing constraints of α, there exists a unique value for the leaders of these equations, (axiom 13_n). For each of these values and for each instantiation of the variables which are not leader in equations of the tree section of α by values which respect the typing constraints of α, there exists a unique value for the leaders of the equations of the tree section of α (axiom 3). Note that the instantiations are conditioned by the fact that they must respect the typing constraints of α, which explains the meaning of the implication in this property. □

Example 5.2.2.2 *Let x, y, z, v, w be variables such that $x \succ y \succ z \succ v \succ w$. We have*

$$T \models num\, w \rightarrow \exists! vxzy \begin{bmatrix} x = fxyw \wedge y = x \wedge \\ 2.z + 2.w = 1 \wedge 3.v + w = 0.1 \wedge \\ tree\, x \wedge num\, v \wedge num\, w \wedge \\ tree\, y \wedge num\, z \end{bmatrix}$$

This property can be written using Property 5.1.1.1 in the following form

$$T \models num\, w \rightarrow \exists! vxzy \begin{bmatrix} x = fxyw \wedge y = x \wedge \\ 2.z = 1 + (-2).w \wedge 3.v = (-1).w \wedge \\ tree\, x \wedge num\, v \wedge num\, w \wedge \\ tree\, y \wedge num\, z \end{bmatrix}$$

for each instantiation of w by numeric values, there exists one and only one value for v and z (axiom 13_n), and thus one and only one value for x and y (axiom 3). Note that if the variable w is instantiated by tree values for example $f(g(0))$, then this formula is equivalent to false in every model of T^*_{ad} according to axioms 5 and 21 and thus

$$T^*_{ad} \not\models \exists! vxzy \begin{bmatrix} x = fxyw \wedge y = x \wedge \\ z + 2.w = 1 \wedge v + w = 0 \wedge \\ tree\, x \wedge num\, v \wedge num\, w \wedge \\ tree\, y \wedge num\, z \end{bmatrix}.$$

Property 5.2.2.3 *Let α and β be two solved blocks having the same numeric section and the same typing constraints. Let α_t and β_t be the tree sections of α and β. If $T^*_{ad} \models \alpha \rightarrow \beta$ and if α_t and β_t have the same set of variables which occur in a left hand side of an equation, then $T^*_{ad} \models \alpha \leftrightarrow \beta$.*

Proof. Since α and β have the same numeric section and the same typing constraints, then there exists a formula δ such that $\alpha = \delta \wedge \alpha_t$ and $\beta = \delta \wedge \beta_t$.

Let $\bar{x}$ be the vector of the variables which occur in a left hand side of an equation of α_t (thus of β_t also) and let X be the set of the variables of $\bar{x}$. Since α and β are solved blocks, then $X \cap var(\delta) = \emptyset$. Let γ be the conjunction of the typing constraints of α (thus of β also) concerning variables which belong to $\bar{x}$. Thus, the formula γ is a sub-formula of δ. According to Property 5.2.2.1, we have $T^*_{ad} \models \gamma \rightarrow \exists! \bar{x} \alpha_t$ and $T^*_{ad} \models \gamma \rightarrow \exists! \bar{x} \beta_t$. Thus, $T^*_{ad} \models \delta \rightarrow \exists! \bar{x}\, \alpha_t$ and $T^*_{ad} \models \delta \rightarrow \exists! \bar{x}\, \beta_t$.

Knowing $T^*_{ad} \models \alpha \rightarrow \beta$, i.e. $T^*_{ad} \models \forall \bar{y} \forall \bar{x}\, \delta \wedge \alpha_t \rightarrow \delta \wedge \beta_t$, with $\bar{y}\bar{x}$ the vector of the variables of $\alpha \wedge \beta$, then the following equivalences are true in T^*_{ad}

$$\begin{array}{lll} & \forall \bar{y} \forall \bar{x}\, \delta \wedge \alpha_t \rightarrow \delta \wedge \beta_t, & \\ \leftrightarrow & \forall \bar{y} \neg(\exists \bar{x}\, \delta \wedge \alpha_t \wedge \neg(\delta \wedge \beta_t)), & \\ \leftrightarrow & \forall \bar{y} \neg(\delta \wedge (\exists \bar{x}\, \alpha_t \wedge \neg \beta_t)), & \text{because } X \cap var(\delta) = \emptyset \\ \leftrightarrow & \forall \bar{y} \neg(\delta \wedge \neg(\exists \bar{x}\, \alpha_t \wedge \beta_t)) & \text{because } T^*_{ad} \models \delta \rightarrow \exists! \bar{x}\, \alpha_t \\ & & \text{ and using corollary 3.1.0.8 (chapter 3),} \\ \leftrightarrow & \forall \bar{y} \neg(\delta \wedge \neg(\exists \bar{x}\, \beta_t \wedge \alpha_t)), & \\ \leftrightarrow & \forall \bar{y} \neg(\delta \wedge (\exists \bar{x}\, \beta_t \wedge \neg \alpha_t)), & \text{because } T^*_{ad} \models \delta \rightarrow \exists! \bar{x}\, \beta_t \\ & & \text{and using corollary 3.1.0.8,} \\ \leftrightarrow & \forall \bar{y} \neg(\exists \bar{x}\, \delta \wedge \beta_t \wedge \neg(\delta \wedge \alpha_t)), & \\ \leftrightarrow & \forall \bar{y} \forall \bar{x}\, \delta \wedge \beta_t \rightarrow \delta \wedge \alpha_t. & \end{array}$$

Thus, we have $T^*_{ad} \models (\alpha \rightarrow \beta) \leftrightarrow (\beta \rightarrow \alpha)$. Since $T^*_{ad} \models \alpha \rightarrow \beta$, we have $T^*_{ad} \models \alpha \leftrightarrow \beta$. □

5.2.3 Decomposition of quantified blocks in T_{ad}^*

Let ψ be a formula. Let $\bar{x}$ be a vector of variables and α a solved block such that for all unreachable quantified variable u in $\exists\bar{x}\alpha$ and all reachable quantified variable v in $\exists\bar{x}\alpha$ we have $u \succ v$. We call *decomposition* of the formula $\exists\bar{x}\alpha \wedge \psi$ the formula

$$\exists\bar{x}^1\,\alpha^1 \wedge (\exists\bar{x}^2\,\alpha^2 \wedge (\exists\bar{x}^3\,\alpha^3 \wedge \psi))), \tag{5.1}$$

obtained as follows : Let X be the set of the variables in $\bar{x}$. Let us decompose the set X into two disjoint subsets: X_r (the set of the elements of X which are reachable in $\exists\bar{x}\alpha$) and X_u. Let *Lead* be the set of the leaders of the equations of α. We have:
− $\bar{x}^1$ is the vector of the variables of X_r.
− $\bar{x}^2$ is the vector of the variables of $X_u - Lead$.
− $\bar{x}^3$ is the vector of the variables of $X_u \cap Lead$.
− α^1 is of the form $\alpha_1^1 \wedge \alpha_2^1$ where α_1^1 is the conjunction of all the equations in $\exists\bar{x}\alpha$ whose leader is reachable, α_2^1 is the conjunction of all the typing constraints of α which concern variables of $var(\alpha_1^1)$.
− α^2 is of the form $\alpha_1^2 \wedge \alpha_2^2$ where α_1^2 is the conjunction of all the inequations of α and α_2^2 is the conjunction of all the typing constraints of α which do not concern variables of $\bar{x}^3$.
− α^3 is of the form $\alpha_1^3 \wedge \alpha_2^3$ where α_1^3 is the conjunction of the other equations and α_2^3 is the conjunction of all the typing constraints of α which concern the variables of $var(\alpha_1^3)$. The restriction on the order $\succ$ of the quantified unreachable and reachable variables is due to an aim to get as leaders of the equations of the numeric section of α unreachable variables. If one quantified leader is reachable then we deduce that all the quantified variables of this equation are reachable.

Let A be the set of the solved blocks. Let A^1 be the set of the formulas of the form $\exists\bar{x}^1\alpha^1$, where α^1 is a solved equation block and all the variables of $\bar{x}^1$ are reachable in $\exists\bar{x}^1\alpha^1$. Let A^2 be the set of solved relation blocks

Property 5.2.3.1 *Let $\exists\bar{x}^1\alpha^1$ be a formula without free variables which belongs to A^1. We have $\bar{x}^1 = \varepsilon$ and $\alpha^1 = true$.*

Proof. Since the formula $\exists\bar{x}^1\alpha^1$ has no free variables, then there exists no reachable variables in $\exists\bar{x}^1\alpha^1$. Thus, according to the definition of the set A^1 we have $\bar{x}^1 = \varepsilon$. Thus, the formula $\exists\bar{x}^1\alpha^1$ is equivalent in T_{ad}^* to the formula without free variables α^1. According to the definition of the set A^1, the formula α^1 is a solved block. Since this solved block does not contain free variables then it is the formula *true*. □

In Chapter 4, we have shown that T_{ad}^* is zero-infinite-decomposable. Thus, we have the following properties

Property 5.2.3.2 *If $\exists\bar{x}^1\alpha^1 \in A^1$ then $T_{ad}^* \models \exists?\bar{x}^1\,\alpha^1$ and for every free variable y in $\exists\bar{x}^1\alpha^1$, at least one of the following properties holds:*

- $T_{ad}^* \models \exists?y\bar{x}^1\,\alpha^1$,
- *there exists a formula $\psi(u) \in \Psi(u)$ such that $T_{ad}^* \models \forall y\,(\exists\bar{x}^1\,\alpha^1) \rightarrow \psi(y)$,*

Property 5.2.3.3 *If $\alpha^2 \in A^2$, then for every x^2 we have $T_{ad}^* \models \exists_{o\,\infty}^{\Psi(u)} x^2\,\alpha^2$.*

Property 5.2.3.4 *If $\alpha^2 \in A^2$, then for every x^2, the formula $\exists x^2\alpha^2$ is equivalent in T_{ad}^* to a formula which belongs to A^2.*

Property 5.2.3.5 *For every decomposed formula of the form (5.1) we have:* $\exists \bar{x}^1 \alpha^1 \in A^1$, $\alpha^2 \in A^2$, $\alpha^3 \in A$ *and* $T^*_{ad} \models \forall \bar{x}^2\, \alpha^2 \rightarrow \exists! \bar{x}^3 \alpha^3$.

Proof. Let $\exists \bar{x}^1 \alpha^1 \wedge (\exists \bar{x}^2 \alpha^2 \wedge (\exists \bar{x}^3 \alpha^3 \wedge \phi))$ be the decomposed formula of $\exists \bar{x} \alpha \wedge \phi$. According to the construction of the sets $\bar{x}^1$, $\bar{x}^2$, $\bar{x}^3$, α^1, α^2 and α^3 defined in Definition 5.2.3, it is clear that $\exists \bar{x}^1 \alpha^1 \in A^1$ and $\alpha^2 \in A^2$. Let us show now that $\forall \bar{x}^2 \alpha^2 \rightarrow \exists! \bar{x}^3 \alpha^3$. Since α^2 contains the typing constraints of all the variables which occur in α and do not occur in $\bar{x}^3$ and since $\bar{x}^3$ contains the leaders of the equations of α^3, then according to Property 5.2.2.1 we have $T^*_{ad} \models \alpha^2 \rightarrow \exists! \bar{x}^3 \alpha^3$, i.e. $\forall \bar{x}^2\, \alpha^2 \rightarrow \exists! \bar{x}^3 \alpha^3$ □

Example 5.2.3.6 *Let v, w, x, y, z be variables such that $w \succ y \succ z \succ x \succ v$. Let us decompose the following formula in T^*_{ad}*

$$\begin{bmatrix} \exists wxyz \\ v = fvx \wedge w + 2.x + (-2).z = 1 \wedge y + 3.z = 0.1 \wedge \\ z < 1 \wedge 3.z + 2.x < 0.1 \wedge \\ \mathit{tree}\, v \wedge \mathit{num}\, w \wedge \mathit{num}\, x \wedge \mathit{num}\, y \wedge \mathit{num}\, z \end{bmatrix} \tag{5.2}$$

the reachable variables of this formula are v and x. Thus, we have $X_{acc} = \{v, x\}$, $X_{inacc} = \{w, y, z\}$ and $Lead = \{v, w, y\}$. Since $w \succ y \succ z \succ x$, then (5.2) is equivalent to the following decomposed formula

$$\begin{bmatrix} \exists x\, v = fvx \wedge \mathit{tree}\, v \wedge \mathit{num}\, x \wedge \\ \begin{bmatrix} \exists z\, z < 1 \wedge 3.z + 2.x < 0.1 \wedge \mathit{num}\, z \wedge \mathit{num} x \wedge \mathit{tree}\, v \wedge \\ \begin{bmatrix} \exists wy\, w + 2.x + (-2).z = 1 \wedge y + 3.z = 0.1 \wedge \\ \mathit{num}\, w \wedge \mathit{num}\, x \wedge \mathit{num}\, y \wedge \mathit{num}\, z) \end{bmatrix} \end{bmatrix} \end{bmatrix}.$$

Note that the elements of A^1 does not accept elimination of quantifiers, since the variables of $\bar{x}^1$ are reachable in $\exists \bar{x}^1\, \alpha^1$. In fact, in the formula $\exists x\, v = fvx$ the quantification $\exists x$ can not be eliminated in T^*_{ad}.

In all what follows we will use the notations $\bar{x}^1$, $\bar{x}^2$, $\bar{x}^3$, α^1, α^2,α^3 to refer to the decomposition of the formula $\exists \bar{x} \alpha$.

5.3 Solving first order constraints in T^*_{ad}

5.3.1 Working formulas and solved formulas

Definition 5.3.1.1 *A normalized formula φ of depth $d \geq 1$ is a formula of the form*

$$\neg(\exists \bar{x}\, \alpha \wedge \bigwedge_{i \in I} \varphi_i), \tag{5.3}$$

with I a finite possibly empty set, α a basic formula and the φ_i normalized formulas of depth d_i and $d = 1 + \max\{0, d_1, ..., d_n\}$.

Of course we have the following property

Property 5.3.1.2 *Every formula is equivalent in T^*_{ad} to a normalized formula.*

Definition 5.3.1.3 *A working formula is a normalized formula in which all the occurrences of $\neg$ are of the form $\neg^k$ with $k \in \{0, ..., 9\}$ and such that each occurrence of a sub-formula of the form*

$$\phi = \neg^k(\exists \bar{x}\, \alpha^c \wedge \alpha^p \wedge \bigwedge_{i \in I} \varphi_i), \tag{5.4}$$

has $\alpha^p = true$ if $k = 0$ and **satisfies the first k conditions** *of the following condition list if $k > 0$. Here α^p is a solved block and is called propagated constraint section, α^c is a basic formula and is called core constraint section, the φ_i are working formulas, and in the conditions: $\beta^p \wedge \beta^c$ is the conjunction of the equations and relations of the immediate top-working formula ψ of ϕ if it exists, i.e. $\psi = \neg^k(\exists \bar{y} \beta^c \wedge \beta^p \wedge \phi \wedge \bigwedge_{j \in J} \phi_j)$ with ϕ_j working formulas.*

1. *if ψ exists then $T \models \alpha^p \wedge \alpha^c \rightarrow \beta^p \wedge \beta^c$, and the tree-sections of α^p and $\beta^c \wedge \beta^p$ have the same set of left-hand side of equations,*
2. *the tree-section of $\alpha^p \wedge \alpha^c$ is formatted and the formula $\alpha^p \wedge \alpha^c$ does not contain $\neg num x \wedge num x$ for any variable x,*
3. *$\alpha^p \wedge \alpha^c$ is a block,*
4. *the numeric-section of $\alpha^p \wedge \alpha^c$ is consistent, and we have $u \succ v$ for u any unreachable variable in $\bar{x}$ and v any reachable variable in $\bar{x}$,*
5. *$\alpha^p \wedge \alpha^c$ is a solved block,*
6. *α^p is the formula $\beta^c \wedge \beta^p$ if ψ exists, and is the formula true otherwise. The formula α^c is a solved block and for each relation $num x$ (or $\neg num x$) in α^p, if x does not occur in an equation or inequation of α^c then $num x$ (resp. $\neg num x$) does not occur in α^c,*
7. *$(\exists \bar{x}\, \alpha^c)$ is decomposable into $(\exists \bar{x}^1\, \alpha^{c1} \wedge (\exists \bar{x}^2\, \alpha^{c2} \wedge (\exists \varepsilon\, true)))$,*
8. *$(\exists \bar{x}\, \alpha^c)$ is decomposable into $(\exists \bar{x}^1\, \alpha^{c1} \wedge (\exists \varepsilon\, \alpha^{c2} \wedge (\exists \varepsilon\, true)))$,*
9. *$(\exists \bar{x}\, \alpha^c)$ is decomposable into $(\exists \bar{x}^1\, \alpha^{c1} \wedge (\exists \varepsilon\, true \wedge (\exists \varepsilon\, true)))$.*

The intuitions behind this definition come from an aim to be able to control the execution of our rewriting rules according to each value of k in a working formula. We strongly insist in the fact that $\neg^k$ does not mean that the normalized formula satisfies only the k^{th} condition but all the conditions i with $1 \leq i \leq k$.

We call *initial* working formula a working formula of the form

$$\neg^6(\exists \varepsilon\, true \wedge \bigwedge_{i \in I} \varphi_i)$$

with φ_i working formulas where all negation symbols $\neg^k$ have $k = 0$ and all propagated constraint sections are reduced to the formula *true*. We call *final* working formula a formula of the form

$$\neg^7(\exists \varepsilon\, true \wedge \bigwedge_{i \in I} \neg^8(\exists \bar{x}_i\, \alpha_i^c \wedge \alpha_i^p \wedge \bigwedge_{j \in J_i} \neg^9(\exists \bar{y}_{ij}\, \beta_{ij}^c \wedge \beta_{ij}^p))), \tag{5.5}$$

where the β_{ij}^c are different from the formula *true*.

Definition 5.3.1.4 *A* general solved formula *is a formula of the form*

$$\exists \bar{x}^1\, \alpha^1 \wedge \alpha^2 \wedge \bigwedge_{i\in I} \neg(\exists \bar{y}^1_i\, \beta^1_i), \tag{5.6}$$

where $\exists \bar{x}^1\, \alpha^1 \in A^1$, $\alpha^2 \in A^2$, $\exists \bar{y}^1_i\, \beta^1_i \in A^1$, *all the* $\alpha^1 \wedge \alpha^2 \wedge \beta^1_i$ *are solved blocks and all the* β^1_i *are different from true.*

According to the properties of $\neg^8$ and $\neg^9$, in the final working formula (5.5), $\alpha^p_i = true$ and $\beta^p_{ij} = \alpha^p_i \wedge \alpha^c_i$. Thus, the formula (5.5) is equivalent in T to the following disjunction of general solved formulas

$$\bigvee_{i\in I} (\exists \bar{x}_i\, \alpha^c_i \wedge \bigwedge_{j\in J_i} \neg(\exists \bar{y}_{ij}\, \beta^c_{ij})) \tag{5.7}$$

Thus, we have the following property

Property 5.3.1.5 *Every working final formula of the form (5.5) is equivalent in* T^*_{ad} *to a disjunction (5.7) of general solved formulas.*

Property 5.3.1.6 *Let* φ *be a working formula of the form*

$$\neg^k(\exists \bar{x}\, \alpha^c \wedge \alpha^p \wedge \bigwedge_{i\in I} \phi_i)$$

with $6 \leq k \leq 9$ *and* ϕ_i *working formulas. We have*

$$T \models \neg(\exists \bar{x}\, \alpha^c \wedge \alpha^p \wedge \bigwedge_{i\in I} \phi^*_i) \leftrightarrow \neg(\alpha^p \wedge (\exists \bar{x}\, \alpha^c \wedge \bigwedge_{i\in I} \phi^*_i))$$

with ϕ^*_i *the normalized formula obtained from* ϕ_i *by replacing all* $\neg^k$ *by* $\neg$.

Proof Let ψ - if it exists - be the immediate top-working formula of φ. Thus, ψ is of the form

$$\neg^k(\exists \bar{y} \beta^c \wedge \beta^p \wedge \varphi \wedge \bigwedge_{j\in J} \phi'_j)$$

with ϕ'_j working formulas. According to Definition 5.3.1.3, since $k \geq 6$ then the normalized formula satisfies the k first conditions of this definition and thus according to the sixth point of this definition we have two cases:

(1) if ϕ does not exists, then α^p is the formula *true* according to the sixth condition of Definition 5.3.1.3. Thus, the property is true.

(2) if ϕ exists, then $\alpha^p = \beta^p \wedge \beta^c$ according to the sixth condition of Definition 5.3.1.3. Since the variables of $\bar{x}$ can not occur in $\beta^c \wedge \beta^p$, then these variables can not occur in α^p, thus we can lift the formula α^p before the quantification $\exists \bar{x}$ and thus the property is true.

Let us present now one of the most important property in T^*_{ad} which shows the differences between the decision procedures defined in Chapter 2 and 3 and an algorithm solving general first order constraints in T^*_{ad}.

Property 5.3.1.7 *Let* φ *be a general solved formula of the form (5.6). If* φ *has no free variables then* φ *is the formula true, otherwise neither* $T \models \varphi$ *nor* $T \models \neg\varphi$

Proof.

Let φ be a general solved formula of the form

$$\exists \bar{x}^1\, \alpha^1 \wedge \alpha^2 \wedge \bigwedge_{i\in I} \neg(\exists \bar{y}^1_i\, \beta^1_i), \tag{5.8}$$

where $\exists \bar{x}^1\, \alpha^1 \in A^1$, $\alpha^2 \in A^2$, $\exists \bar{y}^1_i\, \beta^1_i \in A^1$, all the $\alpha^1 \wedge \alpha^2 \wedge \beta^1_i$ are solved blocks and all the β^1_i are different from *true*. Two cases arise:

Case 1: Let us show that if φ has no free variables then φ is the formula *true*. Since φ has no free variables then $\exists \bar{x}^1\, \alpha^1 \wedge \alpha^2$ has no free variables. Since $\exists \bar{x}^1\, \alpha^1 \in A^1$ and has no free variables then according to Property 5.2.3.1 the formula (5.8) is equivalent in T to the following formula without free variables

$$\alpha^2 \wedge \bigwedge_{i\in I} \neg(\exists \bar{y}^1_i\, \beta^1_i), \tag{5.9}$$

Since $\alpha^2 \in A^2$ and α^2 has no free variables then according to the definition of the set A^2 we have $\alpha^2 = true$. Thus, the preceding formula is equivalent in T^*_{ad} to the following formula without free variables

$$\bigwedge_{i\in I} \neg(\exists \bar{y}^1_i\, \beta^1_i). \tag{5.10}$$

Since $\exists \bar{y}^1_i\, \beta^1_i \in A^1$ and has no free variables, then using Property 5.2.3.1, we deduce that $\exists \bar{y}^1_i \beta^1_i = \exists \varepsilon true$. But according to the condition of the formula (5.8) all the formulas β^1_i are different from *true* and thus I must be the empty set. Thus, the preceding formula is equivalent to *true* in T^*_{ad}.

Case 2: if φ has at least one free variable, then let us show that there exists at least a model M of T^*_{ad} and two distinct instantiations φ' and φ'' of φ by individuals of M such that

$$M \models \neg\varphi' \quad and \quad M \models \varphi''.$$

Let us choose for example for M the standard model of T^*_{ad} given in Chapter 4.[24]

(1) Let us show that φ' exists. Let z be a free variable of φ:

- If z occurs in the formula $\alpha^1 \wedge \alpha^2$ then since $\exists \bar{x}^1\, \alpha^1 \in A^1$ and $\alpha^2 \in A^2$, the formulas α^1 and α^2 are solved blocks, thus all the variables are typed, thus $num z$ or $\neg num z$ is a sub formula of $\alpha^1 \wedge \alpha^2$. To make false φ' it is enough to instantiate the free variable z by an element of Q if $\neg num z$ is a sub formula of $\alpha^1 \wedge \alpha^2$; and by h if $num z$ is a sub formula of $\alpha^1 \wedge \alpha^2$ with $h \in F - \{0,1\}$ a 0-ary function symbol, i.e. a tree constant. By this instantiation φ', we make a contradiction in the typing of z, thus $M \models \neg\varphi'$.
- Else, there exists $k \in I$ such that the formula $\exists \bar{y}^1_k\, \beta^1_k$ with $k \in I$ has at least one free variable. Since $\exists \bar{y}^1_k\, \beta^1_k \in A^1$, then β^1_k is a solved block then according to Property 5.2.1.2 there exists an instantiation $\exists \bar{y}^1_k\, \beta'^1_k$ of the free variables of $\exists \bar{y}^1_k\, \beta^1_k$ such that $M \models \exists \bar{y}^1_k\, \beta'^1_k$ thus $M \models \neg(\exists \bar{x}^1\, \alpha^1 \wedge \alpha^2 \wedge \bigwedge_{i\in I} \neg(\exists \bar{y}^1_i\, \beta^1_i))$, thus $M \models \neg\varphi'$.

(2) Let us show now that there exists φ'' such that $M \models \varphi''$. The formula φ is of the form

$$\exists \bar{x}^1\, \alpha^1 \wedge \alpha^2 \wedge \bigwedge_{i\in I} \neg(\exists \bar{y}^1_i\, \beta^1_i), \tag{5.11}$$

where $\exists \bar{x}^1\, \alpha^1 \in A^1$, $\alpha^2 \in A^2$, $\exists \bar{y}^1_i\, \beta^1_i \in A^1$, all the $\alpha^1 \wedge \alpha^2 \wedge \beta^1_i$ are solved blocks and all the β^1_i are different from *true*.

[24] This model has as domain the set of finite or infinite trees labeled by $Q \cup F$ and such that each sub-tree labeled by $Q \cup \{+,-\}$ is evaluated in Q and reduced to a leaf labeled by Q.

Let α^{2*} be the formula α^2 in which we have removed the typing constraints which concern the leaders of the equations of α^1. Let us also transform the equations of the numeric section of α^1 and β^1_i by moving to the right hand sides the terms containing variables that are not leaders (see Property 5.1.1.1). The preceding formula is equivalent in T^*_{ad} to

$$\exists \bar{x}^1\, \alpha^1 \wedge \alpha^{2*} \wedge \bigwedge_{i \in I} \neg(\exists \bar{y}^1_i\, \beta^1_i), \tag{5.12}$$

where the equations of the numeric section of α^1 (respectively β^1_i) have distinct left hand sides which have no occurrences in other right hand sides of equations of the numeric section of α^1 (respectively β^1_i). This is due to the fact that $\exists \bar{x}^1 \alpha^1 \in A^1$ and $\exists \bar{y}^1_i \beta^1_i \in A^1$, and thus α^1 and β^1_i are solved. Since $\exists \bar{y}^1_i\, \beta^1_i \in A^1$ then β^1_i is a solved block, thus it is consistent and different from *false*. Moreover since β^1_i are different from *true* then each β^1_i has at least a variable. According to the definition of A^1 all the variables of $\bar{y}^1_i$ are reachable and thus there exists at least one free variable in each β^1_i according to the definition of the reachable variables. Since $\alpha^1 \wedge \alpha^{2*} \wedge \beta^1_i$ are solved blocks then they are consistent and thus there exists an instantiation of $\exists \bar{x}^1\, \alpha^1 \wedge \alpha^{2*}$ such that this instantiated formula is true in M (Property 5.2.1.2), thus according to Property 5.2.3.3 there exists an infinity of instantiations of the variables of α^{2*} which make it true in M (and not zero because there exists at least an instantiation since the blocks are solved). For each value of these instantiations and for all instantiations of the right hand sides of the equations of the numeric section of α^1, there exists a value for the leaders of these equations because the leaders of the equations of α^1 do not occur in α^{2*} ($\alpha^1 \wedge \alpha^2 \wedge \beta^1_i$ are solved blocks). For each of these values and instantiations of the variables of the equations of the tree section of α^1 which are not leaders, there exists a value for the leaders of these equations (axiom 3). Then, there exists an infinity of instantiations of the free variables of $\exists \bar{x}^1\, \alpha^1 \wedge \alpha^2$ which make the instantiated formula true in M. Let us show now that there exists from this infinity of instantiations, an instantiation which makes false each formula of the form $\exists \bar{y}^1_i \beta^1_i$ and thus makes true φ''. In each sub-formula of the form $\exists \bar{y}^1_i \beta^1_i$ the leaders of the equations of the numeric section of β^1_i do not occur in the equations and inequations of $\alpha^1 \wedge \alpha^2$ because $\alpha^1 \wedge \alpha^2 \wedge \beta^1_i$ are solved blocks. Since for each instantiation of the right hand sides of the equations of the numeric section of β^1_i there exists a value for the leaders. Thus, it is enough to choose a different value to these leaders to make false all the $\exists \bar{y}^1_i \beta^1_i$. This is possible because the domain M is infinite and more exactly Q is infinite. For each instantiation of the variables which are not leaders in the tree section of β^1_i there exists a unique value for the leaders, thus it is enough to take another value to make false all the $\exists \bar{y}^1_i \beta^1_i$. This is possible because the domain of the trees is infinite and more exactly the set of the function symbols of F is infinite. Thus there exists an instantiation which make true $\exists \bar{x}^1 \alpha^1 \wedge \alpha^2$ and false each sub-formula of the form $\exists \bar{y}^1_i \beta^1_i$. Thus, this instantiation is the formula φ''.

Property 5.3.1.8 *Every general solved formula is equivalent in T to a boolean combination of formulas of the form $\exists \bar{x}^1 \alpha^1 \wedge \alpha^2$, with $\exists \bar{x}^1 \alpha^1 \in A^1$ and $\alpha^2 \in A^2$, which do not accept elimination of quantifiers.*

Proof. Let φ be the following general solved formula

$$\bigvee_{i \in I} (\exists \bar{x}_i \alpha^c_i \wedge \bigwedge_{j \in J_i} \neg(\exists \bar{y}_{ij} \beta^c_{ij})) \tag{5.13}$$

where the β^c_{ij} are different from *true*. The formula φ is extracted from a final working formula of the form

$$\neg^7(\exists\epsilon true \bigwedge_{i\in I} \neg^8(\exists\bar{x}_i\alpha_i^c \wedge \alpha_i^p \wedge \bigwedge_{j\in J_i} \neg^9(\exists\bar{y}_{ij}\beta_{ij}^c \wedge \beta_{ij}^p))$$

According to the conditions of $\neg^8$, we have $\alpha_i^p = true$ and all the variables of $\bar{x}_i$ are reachable in $\exists\bar{x}_i\alpha_i^c$. Moreover, $\exists\bar{x}_i\alpha_i^c$ is decomposed in T^*_{ad} into $\exists\bar{x}_i\alpha_i^{c1}\wedge\alpha_i^{c2}$, with $\exists\bar{x}_i\alpha c1_i \in A^1$ and $\alpha_i^{c2} \in A^2$.

According to the conditions of $\neg^9$, we have $\beta_{ij}^p = \alpha_i^c \wedge \alpha_i^p = \alpha_i^c$, the $\beta_{ij}^c \wedge \beta_{ij}^p$ are solved blocks and the $\exists\bar{y}_{ij}\beta_{ij}^c$ belong to A^1. Thus, we deduce that $\beta_{ij}^c\wedge\alpha_i^c$ are solved blocks. Since each variable in $\bar{x}_i$ is reachable in $\exists\bar{x}_i\alpha_i^c$, it remains reachable in $\exists\bar{x}_i\bar{y}_{ij}\alpha_i^c \wedge \beta_{ij}^c$. Since each variable y in $\bar{y}_{ij}$ is reachable in $\exists\bar{y}_{ij}\beta_{ij}^c$, two cases arise: (1) y is reachable without using variables in $\bar{x}_i$, in this case, y remains reachable in $\exists\bar{x}_i\bar{y}_{ij}\alpha_i^c \wedge \beta_{ij}^c$, (2) y is reachable using variables in $\bar{x}_i$, in this case, since all variables in $\bar{x}_i$ are reachable in $\exists\bar{x}_i\bar{y}_{ij}\alpha_i^c \wedge \beta_{ij}^c$, then y is still reachable in this formula.

Thus, the formulas $\exists\bar{x}_i\bar{y}_{ij}\alpha_i^c\wedge\beta_{ij}^c$ can be decomposed into $\exists\bar{x}_i\bar{y}_{ij}\alpha_i^{c1}\wedge\beta_{ij}^c\wedge\alpha_i^{c2}$, with $\exists\bar{x}_i\bar{y}_{ij}\alpha_i^{c1}\wedge \beta_{ij}^c \in A^1$ and $\alpha_i^{c2} \in A^2$.

According to Property 5.2.1.1, the formula (5.13) is equivalent in T to the formula

$$\bigvee_{i\in I}((\exists\bar{x}_i\alpha_i^c) \wedge \bigwedge_{j\in J_i} \neg(\exists\bar{x}_i\bar{y}_{ij}\alpha_i^c \wedge \beta_{ij}^c))$$

We have proved that each quantified conjunction is of the form $\exists\bar{x}^1\alpha^1 \wedge \alpha^2$ where $\bar{x}^1\alpha^1 \in A^1$ and $\alpha^2 \in A^2$. This property is then proved.□

5.3.2 Main idea

The general algorithm for solving first-order constraints in T uses a system of rewriting rules. The main idea is to transform an initial working formula of depth d into a final working formula of depth less than or equal to three. The transformation is done in two steps:

(1) The first step is a top-down simplification and propagation. In each sub-working formula, $\alpha^c \wedge \alpha^p$ is transformed into a solved block, then $\exists\bar{x}\alpha^c$ is decomposed into three parts. The third part is eliminated and added to the core-constraint section of the immediate sub-working formulas using a special property of the quantifier $\exists!$. The constraints of the two other parts in α^p are propagated to the propagated-constraint section of the immediate sub-working formulas. In this step, the rules 1 to 24 are applied and transform the initial working formula into a working formula where each negation symbol is of the form $\neg^7$.

(2) The second step is a bottom-up simplification and elimination of quantifiers. This step is done by the rules 25 to 28. In each sub-working formula of depth one or two, the rule 25 eliminates quantified variables of the second part of the decomposition (the third one had been already removed in the first step). The rule 26 eliminates the constraints of the second part in the deepest level. Each sub-working formula of depth 3 is transformed step by step to a conjunction of working formulas of depth 2 by the rule 28 using a property of the quantifier $\exists?$. The transformations in this step can create new sub-working formulas where the first step needs to be done. At the end of the transformations, we obtain a final working formula of depth less than or equal to 3.

5.3.3 The rewriting rules

We present now the rewriting rules which transform an initial working formula to a final working formula, which is equivalent in T^*_{ad}. To apply the rule $p_1 \Longrightarrow p_2$ to the working formula p means to replace in p, a sub-formula p_1 by the formula p_2, by considering that the connector $\wedge$ is associative and commutative.

1	$\neg^1(\exists\bar{u}\, num\, x \wedge tree\, x \wedge \alpha \wedge \varphi)$	$\Longrightarrow$	$true$
2	$\neg^1(\exists\bar{u}\, x = f\bar{y} \wedge x = g\bar{z} \wedge tree\, x \wedge \alpha \wedge \varphi)$	$\Longrightarrow$	$true$
3	$\neg^1(\exists\bar{u}\, x = x \wedge \alpha \wedge \varphi)$	$\Longrightarrow$	$\neg^1(\exists\bar{u}\, \alpha \wedge \varphi)$
4	$\neg^1(\exists\bar{u}\, y = x \wedge tree\, x \wedge \alpha \wedge \varphi)$	$\Longrightarrow$	$\neg^1(\exists\bar{u}\, x = y \wedge tree\, x \wedge \alpha \wedge \varphi)$
5	$\neg^1\begin{bmatrix}\exists\bar{u}\, x = fy_1...y_n \wedge x = fz_1...z_n\wedge \\ tree\, x \wedge \alpha \wedge \varphi\end{bmatrix}$	$\Longrightarrow$	$\neg^1\begin{bmatrix}\exists\bar{u}\, x = fy_1...y_n \wedge \bigwedge_i y_i = z_i\wedge \\ tree\, x \wedge \alpha \wedge \varphi\end{bmatrix}$
6	$\neg^1\begin{bmatrix}\exists\bar{u}\, x = y \wedge x = fz_1...z_n\wedge \\ tree\, x \wedge tree\, y \wedge \alpha \wedge \varphi\end{bmatrix}$	$\Longrightarrow$	$\neg^1\begin{bmatrix}\exists\bar{u}\, x = y \wedge y = fz_1...z_n\wedge \\ tree\, x \wedge tree\, y \wedge \alpha \wedge \varphi\end{bmatrix}$
7	$\neg^1(\exists\bar{u}\, x = y \wedge x = z \wedge tree\, x \wedge \alpha \wedge \varphi)$	$\Longrightarrow$	$\neg^1(\exists\bar{u}\, x = y \wedge y = z \wedge tree\, x \wedge \alpha \wedge \varphi)$
8	$\neg^4(\exists\bar{u}\, 0 = 0.1 \wedge \alpha \wedge \varphi)$	$\Longrightarrow$	$\neg^4(\exists\bar{u}\, \alpha \wedge \varphi)$
9	$\neg^4(\exists\bar{u}\, 0 < a_0.1 \wedge \alpha \wedge \varphi)$	$\Longrightarrow$	$\neg^4(\exists\bar{u}\, \alpha \wedge \varphi)$
10	$\neg^4\begin{bmatrix}\exists\bar{u}\, x = y\wedge \\ num\, x \wedge num\, y \wedge \alpha \wedge \varphi\end{bmatrix}$	$\Longrightarrow$	$\neg^4\begin{bmatrix}\exists\bar{u}\, x + (-1).y = 0.1\wedge \\ num\, x \wedge num\, y \wedge \alpha \wedge \varphi\end{bmatrix}$
11	$\neg^4\begin{bmatrix}\exists\bar{u}\, x = -y\wedge \\ num\, x \wedge num\, y \wedge \alpha \wedge \varphi\end{bmatrix}$	$\Longrightarrow$	$\neg^4\begin{bmatrix}\exists\bar{u}\, x + y = 0.1\wedge \\ num\, x \wedge num\, y \wedge \alpha \wedge \varphi\end{bmatrix}$
12	$\neg^4\begin{bmatrix}\exists\bar{u}\, x = y + z \wedge num\, x\wedge \\ num\, y \wedge num\, z \wedge \alpha \wedge \varphi\end{bmatrix}$	$\Longrightarrow$	$\neg^4\begin{bmatrix}\exists\bar{u}\, x + (-1).y + (-1).z = 0.1\wedge \\ num\, x \wedge num\, y \wedge num\, z \wedge \alpha \wedge \varphi\end{bmatrix}$
13	$\neg^4\begin{bmatrix}\exists\bar{u}\, \Sigma_{i=1}^n a_i.x_i = a_0.1\wedge \\ \Sigma_{j=1}^n b_j.x_j = b_0.1\wedge \\ \bigwedge num\, x \wedge \alpha \wedge \varphi\end{bmatrix}$	$\Longrightarrow$	$\neg^4\begin{bmatrix}\exists\bar{u}\, \Sigma_{i=1}^n a_i.x_i = a_0.1\wedge \\ \Sigma_{i=1}^n (b_k a_i - a_k b_i).x_i = (b_k a_0 - a_k b_0).1\wedge \\ \bigwedge num\, x \wedge \alpha \wedge \varphi\end{bmatrix}$
14	$\neg^4\begin{bmatrix}\exists\bar{u}\, \Sigma_{i=1}^n a_i.x_i = a_0.1\wedge \\ \Sigma_{j=1}^n b_j.x_j < b_0.1\wedge \\ \bigwedge num\, x \wedge \alpha \wedge \varphi\end{bmatrix}$	$\Longrightarrow$	$\neg^4\begin{bmatrix}\exists\bar{u}\, \Sigma_{i=1}^n a_i.x_i = a_0.1\wedge \\ \Sigma_{i=1}^n \lambda(b_k a_i - a_k b_i).x_i < (b_k a_0 - a_k b_0).1\wedge \\ \bigwedge num\, x \wedge \alpha \wedge \varphi\end{bmatrix}$
15	$\neg^1(\exists\bar{x}\, \alpha^c \wedge \alpha^p \wedge \varphi)$	$\Longrightarrow$	$\neg^2(\exists\bar{x}\, \alpha^c \wedge \alpha^p \wedge \varphi)$
16	$\neg^2(\exists\bar{x}\, \alpha^c \wedge \alpha^p \wedge \varphi)$	$\Longrightarrow$	$\neg^1(\exists\bar{x}\, num\, z \wedge \alpha^c \wedge \alpha^p \wedge \varphi)$
17	$\neg^2(\exists\bar{x}\, \alpha^c \wedge \alpha^p \wedge \varphi)$	$\Longrightarrow$	$\neg^1(\exists\bar{x}\, tree\, z \wedge \alpha^c \wedge \alpha^p \wedge \varphi)$
18	$\neg^2(\exists\bar{x}\, \alpha^c \wedge \alpha^p \wedge \varphi)$	$\Longrightarrow$	$\begin{bmatrix}\neg^1(\exists\bar{x}\, num\, z \wedge \alpha^c \wedge \alpha^p \wedge \varphi)\wedge \\ \neg^1(\exists\bar{x}\, tree\, z \wedge \alpha^c \wedge \alpha^p \wedge \varphi)\end{bmatrix}$
19	$\neg^2(\exists\bar{x}\, \alpha^c \wedge \alpha^p \wedge \varphi)$	$\Longrightarrow$	$\neg^3(\exists\bar{x}\, \alpha^c \wedge \alpha^p \wedge \varphi)$
20	$\neg^3(\exists\bar{x}\, \alpha^c \wedge \alpha^p \wedge \varphi)$	$\Longrightarrow$	$true$
21	$\neg^3(\exists\bar{x}\, \alpha^c \wedge \alpha^p \wedge \varphi)$	$\Longrightarrow$	$\neg^4(\exists\bar{x}\, \alpha^c \wedge \alpha^p \wedge \varphi)$
22	$\neg^4(\exists\bar{x}\, \alpha^c \wedge \alpha^p \wedge \varphi)$	$\Longrightarrow$	$\neg^5(\exists\bar{x}\, \alpha^c \wedge \alpha^p \wedge \varphi)$
23	$\neg^7\begin{bmatrix}\exists\bar{x}\, \alpha^c \wedge \alpha^p \wedge \varphi\wedge \\ \neg^5(\exists\bar{y}\, \beta^c \wedge \beta^p \wedge \psi)\end{bmatrix}$	$\Longrightarrow$	$\neg^7\begin{bmatrix}\exists\bar{x}\, \alpha^c \wedge \alpha^p \wedge \varphi\wedge \\ \neg^6(\exists\bar{y}\, \gamma^c \wedge \gamma^p \wedge \psi)\end{bmatrix}$
24	$\neg^6\begin{bmatrix}\exists\bar{x}\, \alpha^c \wedge \alpha^p\wedge \\ \bigwedge_i \neg^0(\exists\bar{y}_i\beta_i^c \wedge \beta_i^p \wedge \varphi_i)\end{bmatrix}$	$\Longrightarrow$	$\neg^7\begin{bmatrix}\exists\bar{x}^1\bar{x}^2\, \alpha^{c1} \wedge \alpha^{c2} \wedge \alpha^p\wedge \\ \bigwedge_i \neg^1(\exists\bar{y}_i\bar{x}^3\gamma_i^c \wedge \gamma_i^p \wedge \varphi_i)\end{bmatrix}$
25	$\neg^7\begin{bmatrix}\exists\bar{x}\, \alpha^c \wedge \alpha^p\wedge \\ \bigwedge_{i\in I} \neg^9(\exists\bar{y}_i\beta_i^c \wedge \beta_i^p)\end{bmatrix}$	$\Longrightarrow$	$\neg^8\begin{bmatrix}\exists\bar{x}^1\alpha^{c1} \wedge \alpha^{c2*} \wedge \alpha^p\wedge \\ \bigwedge_{i\in I'} \neg^9(\exists\bar{y}_i\beta_i^c \wedge \beta_i^{p*})\end{bmatrix}$
26	$\neg^7\begin{bmatrix}\exists\bar{x}\, \alpha^c \wedge \alpha^p \wedge \varphi\wedge \\ \neg^8(\exists\bar{y}\beta^c \wedge \beta^p)\end{bmatrix}$	$\Longrightarrow$	$\begin{bmatrix}\neg^7(\exists\bar{x}\, \alpha^c \wedge \alpha^p \wedge \varphi \wedge \neg^9(\exists\bar{y}\beta^{c1} \wedge \beta^p))\wedge \\ \bigwedge_{i\in I} \neg^1(\exists\bar{x}\bar{y}\beta^p \wedge \beta^{c1} \wedge \beta_i^{c2*} \wedge \varphi_0)\end{bmatrix}$

$$27 \quad \neg^7 \begin{bmatrix} \exists \bar{x}\, \alpha^c \wedge \alpha^p \wedge \varphi \wedge \\ \neg^9(\exists \varepsilon\, true \wedge \beta^p) \end{bmatrix} \quad \Longrightarrow \quad true$$

$$28 \quad \neg^7 \begin{bmatrix} \exists \bar{x}\, \alpha^c \wedge \alpha^p \wedge \varphi \wedge \\ \neg^8 \begin{bmatrix} \exists \bar{y}\, \beta^c \wedge \beta^p \wedge \\ \bigwedge_{i \in I} \neg^9(\exists \bar{z}_i\, \gamma_i^c \wedge \gamma_i^p) \end{bmatrix} \end{bmatrix} \quad \Longrightarrow \quad \begin{bmatrix} \neg^7(\exists \bar{x}\, \alpha^c \wedge \alpha^p \wedge \varphi \wedge \neg^8(\exists \bar{y}\, \beta^c \wedge \beta^p)) \wedge \\ \bigwedge_{i \in I} \neg^6(\exists \bar{x}\bar{y}\bar{z}_i\, \delta_i^c \wedge \delta_i^p \wedge \varphi_0) \end{bmatrix}$$

In all these rules, α is a basic formula, φ and ψ are conjunctions of working formulas.

In the rules 1 to 14, the equations and relations in α^c and α^p are mixed by considering the connector $\wedge$ associative and commutative. In these rules, except the rule 6, all modifications in the right hand side are done in α^c, since α^p is a solved block.

In the rule 2, f and g are two distinct function symbols taken from F. In the rules 4, 6, 7 $x \succ y$ This condition prevents infinite loops and makes the procedure terminating. In the rule 5, the equation $x = fz_1...z_n$ does not belong to α^p. In the rule 6, if the equation $x = fz_1...z_n$ belongs to α^p, then $x = y \wedge \neg numy$ is moved to α^p. In the rule 7, the equation $x = z$ does not belong to α^p.

We recall that the notation 0.1 in the rule 8 means the term 0. In the rule 9, $a_0 > 0$. In the rules 13 and 14 the variable x_k is the leader of the equation $\Sigma_i a_i x_i = a_0 1$ and $b_k \neq 0$. Moreover the equation $\Sigma_i b_i.x_i = b_0.1$ does not belong to α^p. In the rule 14, the relation $\Sigma_i b_i.x_i < b_0.1$ does not belong to α^p and $\lambda = 1$ if $a_k > 0$ and $\lambda = -1$ otherwise.

In the rule 15, the tree section of $\alpha^c \wedge \alpha^p$ is formatted and there is no sub-formula in $\alpha^c \wedge \alpha^p$ of the form $numx \wedge \neg numx$. In the rule 16 respectively 17, the typing constraint $numz$, respectively $\neg numz$ is not in $\alpha^c \wedge \alpha^p$ and is a consequence of $\alpha^c \wedge \alpha^p$. In the rule 18, z does not have typing constraint in $\alpha^c \wedge \alpha^p$ and neither $numz$ nor $\neg numz$ is a consequence of $\alpha^c \wedge \alpha^p$.

In the rule 19, $\alpha^c \wedge \alpha^p$ is a block. In the rule 20, the numeric section of $\alpha^c \wedge \alpha^p$ is inconsistent. In the rule 21, the unreachable variables in $\bar{x}$ are renamed if necessary such that $u \succ v$ for each unreachable variable u and each reachable variable v in $\bar{x}$ and the numeric section of $\alpha^c \wedge \alpha^p$ is consistent. The consistency can be verified for example by using the first step of the Simplex. In the rule 22, $\alpha^c \wedge \alpha^p$ is a solved block.

In the rule 23, γ^c is obtained from β^c as follows: for all variable $x \in var(\beta^c)$, we add all the relations $numx$ or $\neg numx$ which are in β^p but not in β^c, and for all the variables y which do not occur in an equation or inequation of β^c we remove all relations $numy$ or $\neg numy$ which are both in β^c and β^p. The formula γ^p is the formula $\alpha^p \wedge \alpha^c$.

In the rule 24, $\exists \bar{x}\alpha^c$ is decomposed to $\exists \bar{x}^1 \alpha^{c1} \wedge (\exists \bar{x}^2 \alpha^{c2} \wedge (\exists \bar{x}^3 \alpha^{c3}))$, $\gamma_i^c = \beta_i^c \wedge \alpha^{c3}$ and $\gamma_i^p = \beta_i^p \wedge \alpha^{c1} \wedge \alpha^{c2} \wedge \alpha^p$.

The four rules 25, 26, 27 and 28 can not be applied on the occurrence of $\neg^7$ of the first level of the general working formula. In the rule 25, I' is the set of $i \in I$ such that β_i^c does not contain occurrences of any variables in $\bar{x}^2$. The formula α^{c2*} is such that $T \models (\exists \bar{x}^2 \alpha^{c2}) \leftrightarrow \alpha^{c2*}$ and is computed using the Fourier quantifier elimination. The propagated-constraint section $\beta_i^{p*} = \alpha^{c1} \wedge \alpha^{c2*} \wedge \alpha^p$.

In the rule 26, φ is such that every negation symbol $\neg^k$ has $k \geq 6$, φ_0 is obtained from φ by replacing all occurrences of $\neg^k$ by $\neg^0$ and all propagated-constraint sections by *true*. Let β^2 the formula obtained from β^{c2} by removing the multiple occurrences of typing constraints and for all the variables y which do not occur in an inequation of β^{c2} we remove all relation $numy$ or $\neg numy$ which are both in β^{c1} and β^{c2}. If β^2 is the formula *true* then $I = \emptyset$, otherwise the β_i^{c2*} with $i \in I$ are obtained from β^2 as follows: Since $\beta^2 \in A^2$ then it is of the form

$$\begin{bmatrix} (\bigwedge_{\ell \in L} numz_\ell) \wedge (\bigwedge_{k \in K} \neg numv_k) \wedge \\ ((\bigwedge_{j \in J} \sum_{i=1}^n a_{ij}.x_i < a_{0j}.1) \wedge \bigwedge_{m=1}^n numx_m) \end{bmatrix},$$

thus $\neg\beta^2$ is of the form

$$\begin{bmatrix}(\bigvee_{\ell\in L}\neg num\, z_\ell)\vee(\bigvee_{k\in K} numv_k)\vee(\bigvee_{m=1}^{n}\neg numx_m)\vee\\ \bigvee_{j\in J}((\sum_{i=1}^{n} a_{ij}x_i = a_{0j}.1\wedge\bigwedge_{m=1}^{n} numx_m)\vee\\ (\sum_{i=1}^{n}(-a_{ij})x_i < (-a_{0j}).1\wedge\bigwedge_{m=1}^{n} numx_m))\end{bmatrix}$$

Each element of this disjunction is a block and represents a formula β_i^{c2*}. Of course we have $T \models (\neg\beta^2) \leftrightarrow \bigvee_i \beta_i^{c2*}$.

In the rule 28, $I \neq \emptyset$, φ is such that every negation symbol $\neg^k$ has $k \geq 6$, φ_0 is obtained from φ by replacing all occurrences of $\neg^k$ by $\neg^0$ and all propagated-constraint sections by *true*. Moreover $\delta_i^p = \alpha^p$ and $\delta_i^c = \gamma_i^c \wedge \beta^c \wedge \alpha^c$.

Property 5.3.3.1 *Every repeated application of the preceding rewriting rules on an initial working formula terminates and produces a wnfv final working formula equivalent in T^*_{ad}.*

Proof, first part: Let us show that every repeated application of these rules on an initial working formula terminates. Note that the rules 1...7 are applied on sub-working formulas which begin by the symbol $\neg^1$ without changing the value of this symbol. By the same way, the rules 8 ...14 are applied on sub-working formulas which begin by the symbol $\neg^4$. Let us then divide this proof into three parts: (1) every application of the rules 1...7 on a working formula which begins by $\neg^1$ terminates, (2) every repeated application of the rules 8...14 on a sub-working formula which begins by $\neg^4$ terminates, (3) every repeated application of the rules 15 ... 28 terminates.

(1) Let us show that every application of the rules 1... 7 on a sub-working formula which begins by $\neg^1(\exists\bar{x}\alpha\wedge\varphi)$ terminates. Since the variables of V are ordered by a linear dense order relation $\succ$ without endpoints, then we can associate to each variable x a positive integer $no(x)$, such that $x \succ y$ if $no(x) > no(y)$. Let the 3-tuple (n_1, n_2, n_3) where

- n_1 is the number of equations of the form $x = fy_1...y_n$ in α,
- n_2 is the sum of the $no(x)$ for all occurrences of a variable x in α,
- n_3 is the number of equations of the form $y = x$ with $x \succ y$ in α.

for each rule, there exists a row i such that the application of the rules decreases or does not change the values of n_j, with $1 \leq j < i$, and decreases the value of n_i. This row i is equal to 1 for the rules (2) and (5), 2 for the rules (1), (3), (6) and (7), and 3 for the rule (4). To each sequence of formulas obtained by finite application of the rules, we can associate a series of 3-tuples of the form (n_1, n_2, n_3) which is strictly decreasing in the lexicographic order. Since the n_i are positive integers, they can not be negative and thus this series is finite and the application of the rules 1...7 terminates.

(2) Let us show now that every application of the rules 8...14 on a sub-working formula which begins by $\neg^4$ terminates. This termination is evident since the rules 8 ...12 transform the equations and inequations into a basic form and the rules 13 and 14 remove the double occurrences of the leaders.

(3) Let us show now that every repeated application of the rules 15 ... 28 terminates. Starting with an initial working formula of the form $\neg^6(\exists\varepsilon true\wedge\varphi)$, with φ a conjunction of working formulas where all the negations are of the form $\neg^0$, the rule 24 is the only one that can be applied by changing the $\neg^6$ to $\neg^7$ and all the internal $\neg^0$ to $\neg^1$. According to what we have shown every repeated application of the rules 1... 7 on a sub-working formula which begins by $\neg^1$ terminates. Then, the rule 15 changes $\neg^1$ into $\neg^2$. For every sub-working formula which

begins by $\neg^2$, the rules 16, 17 and 18 can be applied at most one time for each free variable which has not yet a typing constraint in $\alpha^c \wedge \alpha^p$. These rules create new working formulas which begins by $\neg^1$. This loop is finite since we never add new untyped variables during the application of this part of rules. Note also that every application of the rules 19 to 24 terminates. Concerning the rules 25 and 27, they can be applied one and only one time on each sub-working formula. In the rule 26, we replace a sub-working formula containing a sequence of $\neg^7\neg^8$ by the same working formula with a sequence $\neg^7\neg^9$ and containing $|I|$ working formulas where the sequence of $\neg^8$ has been removed. In the rule 28 we decrease the size of the depth of the working formula. Thus, we can not apply infinitely these rules. This is a semi-formal proof ; we can make a better proof using a big n-tuples due to the high number of rules.

Proof, second part: Let us show that the rules are correct in T^*_{ad}. The rules 1..14 are evident in T^*_{ad} and come from the axiomatization of T^*_{ad}. In the rule 15, since the tree section of $\alpha^c \wedge \alpha^p$ is formated and does not contain sub-formulas of the form $\mathit{num}\, x \wedge \mathit{tree}\, x$, the symbol $\neg^1$ can be changed into $\neg^2$. Thus, this rule is correct.

In the rule 16, since $\mathit{num}\, z$ is a consequence of $\alpha^c \wedge \alpha^p$, then the formula $\alpha^c \wedge \alpha^p$ is equivalent in T^*_{ad} to $\alpha^c \wedge \alpha^p \wedge \mathit{num}\, z$. Thus, this rule is correct. By the same way, we show the correctness of the rules 17 and 18.

In the rule 20, since the numeric section of $\alpha^c \wedge \alpha^p$ is inconsistent, then the formula $\alpha^c \wedge \alpha^p$ is equivalent in T^*_{ad} to *false*. Thus, this rule is correct.

The rules 19, 21 and 22 are correct because their conditions are sufficient to change their negation symbols into $\neg^3, \neg^4, \neg^5$ (respectively)

Correctness of the rule 23:

$$\neg^7 \begin{bmatrix} \exists \bar{x}\, \alpha^c \wedge \alpha^p \wedge \varphi \wedge \\ \neg^5(\exists \bar{y}\, \beta^c \wedge \beta^p \wedge \psi) \end{bmatrix} \Longrightarrow \neg^7 \begin{bmatrix} \exists \bar{x}\, \alpha^c \wedge \alpha^p \wedge \varphi \wedge \\ \neg^6(\exists \bar{y}\, \gamma^c \wedge \gamma^p \wedge \psi) \end{bmatrix}$$

where γ^c is obtained from β^c as follows: for all variable $x \in \mathit{var}(\beta^c)$, we add all the relations numx or $\neg \mathit{numx}$ which are in β^p but not in β^c, and for all the variables y which do not occur in an equation or inequation of β^c we remove all relations numy or $\neg \mathit{numy}$ which are both in β^c and β^p. The formula γ^p is the formula $\alpha^p \wedge \alpha^c$.

We know that $\beta^c \wedge \beta^p$ is equivalent to $\gamma^c \wedge \beta^p$ in T^*_{ad}. Thus, let β^p_t be the tree section of β^p and β^p_n the numeric section of β^p. Let α^{cp}_t be the tree section of $\alpha^c \wedge \alpha^p$ and α^{cp}_n the numeric section of $\alpha^c \wedge \alpha^p$. According to the conditions of $\neg^5$, α^{cp}_t and β^p_t have the same set of variables which occur in the left hand sides of equations. We have also $\alpha^{cp}_n = \beta^p_n$ and $T^*_{ad} \models \beta^c \wedge \beta^p \rightarrow \alpha^c \wedge \alpha^p$. Thus

$$T \models \gamma^c \wedge \beta^p \rightarrow \gamma^c \wedge \alpha^c \wedge \alpha^p,$$

i.e.

$$T \models \gamma^c \wedge \beta^p_t \wedge \beta^p_n \rightarrow \gamma^c \wedge \alpha^{cp}_t \wedge \alpha^{cp}_n,$$

and thus

$$T \models \gamma^c \wedge \beta^p_t \wedge \alpha^{cp}_n \rightarrow \gamma^c \wedge \alpha^{cp}_t \wedge \alpha^{cp}_n.$$

Since the tree sections of $\gamma^c \wedge \beta^p_t$ and $\gamma^c \wedge \alpha^{cp}_t$ have the same set of variables which occur in the left hand sides of equation and according to Property 5.2.2.3, we have

$$T \models \gamma^c \wedge \beta^p_t \wedge \alpha^{cp}_n \leftrightarrow \gamma^c \wedge \alpha^{cp}_t \wedge \alpha^{cp}_n,$$

thus

$$T \models \beta^c \wedge \beta^p \leftrightarrow \gamma^c \wedge \alpha^c \wedge \alpha^p.$$

Since $\gamma^p = \alpha^c \wedge \alpha^p$ then the rule 23 is correct in T^*_{ad}.

Correctness of the rule 24:

$$\neg^6 \begin{bmatrix} \exists \bar{x}\, \alpha^c \wedge \alpha^p \wedge \\ \bigwedge_i \neg^0(\exists \bar{y}_i \beta_i^c \wedge \beta_i^p \wedge \varphi_i) \end{bmatrix} \Longrightarrow \neg^7 \begin{bmatrix} \exists \bar{x}^1 \bar{x}^2\, \alpha^{c1} \wedge \alpha^{c2} \wedge \alpha^p \wedge \\ \bigwedge_i \neg^1(\exists \bar{y}_i \bar{x}^3 \gamma_i^c \wedge \gamma_i^p \wedge \varphi_i) \end{bmatrix}$$

with $\gamma_i^c = \beta_i^c \wedge \alpha^{c3}$ and $\gamma_i^p = \beta_i^p \wedge \alpha^{c1} \wedge \alpha^{c2} \wedge \alpha^p$.

According to Definition 5.3.1.3 of working formula, since we have the symbol $\neg^6$ then $\beta_i^p = \alpha^c \wedge \alpha^p$. Thus, the left hand side of this rule is equivalent in T^*_{ad} to a formula of the form

$$\neg(\exists \bar{x}\, \alpha^c \wedge \alpha^p \wedge \bigwedge_i \neg(\alpha^c \wedge \alpha^p \wedge (\exists \bar{y}_i \beta_i^c \wedge \varphi_i))),$$

thus to

$$\neg(\exists \bar{x}\, \alpha^c \wedge \alpha^p \wedge \bigwedge_i \neg(\exists \bar{y}_i \beta_i^c \wedge \varphi_i)).$$

According to Property 5.3.1.6, the preceding formula is equivalent in T^*_{ad} to

$$\neg(\alpha^p \wedge (\exists \bar{x}\, \alpha^c \wedge \bigwedge_i \neg(\exists \bar{y}_i \beta_i^c \wedge \varphi_i))).$$

According to Definition 5.3.1.3 of working formula, since we have $\neg^6$ then the conditions 4,5,6 of Definition 5.3.1.3 hold. Thus, the formula $\exists \bar{x} \alpha^c$ can be decomposed in T^*_{ad}. The preceding formula is thus equivalent in T^*_{ad} to a formula of the form

$$\neg(\alpha^p \wedge (\exists \bar{x}^1 \alpha^{c1} \wedge (\exists \bar{x}^2 \alpha^{c2} \wedge (\exists \bar{x}^3 \alpha^{c3} \wedge \bigwedge_i \neg(\exists \bar{y}_i \beta_i^c \wedge \varphi_i))))),$$

with $T^*_{ad} \models \forall \bar{x}^2 \alpha^{c2} \rightarrow \exists! \bar{x}^3 \alpha^{c3}$. According to Corollary 3.1.0.8 of Chapter 3, the preceding formula is equivalent in T^*_{ad} to

$$\neg(\alpha^p \wedge (\exists \bar{x}^1 \alpha^{c1} \wedge (\exists \bar{x}^2 \alpha^{c2} \wedge \bigwedge_i \neg(\exists \bar{x}^3 \bar{y}_i\, \alpha^{c3} \wedge \beta_i^c \wedge \varphi_i)))),$$

i.e. to

$$\neg \begin{bmatrix} \exists \bar{x}^1 \bar{x}^2\, \alpha^{c1} \wedge \alpha^{c2} \wedge \alpha^p \wedge \\ \bigwedge_i \neg(\exists \bar{x}^3 \bar{y}_i\, \alpha^{c3} \wedge \beta_i^c \wedge \alpha^{c1} \wedge \alpha^{c2} \wedge \alpha^p \wedge \varphi_i) \end{bmatrix}.$$

This rule is thus correct in T^*_{ad}.

Correctness of the rule 25:

$$\neg^7 \begin{bmatrix} \exists \bar{x}\, \alpha^c \wedge \alpha^p \wedge \\ \bigwedge_{i \in I} \neg^9(\exists \bar{y}_i \beta_i^c \wedge \beta_i^p) \end{bmatrix} \Longrightarrow \neg^8 \begin{bmatrix} \exists \bar{x}^1 \alpha^{c1} \wedge \alpha^{c2*} \wedge \alpha^p \wedge \\ \bigwedge_{i \in I'} \neg^9(\exists \bar{y}_i \beta_i^c \wedge \beta_i^{p*}) \end{bmatrix}$$

where I' is the set of $i \in I$ such that β_i^c does not contain occurrences of any variables in $\bar{x}^2$. The formula α^{c2*} is such that $T^*_{ad} \models (\exists \bar{x}^2 \alpha^{c2}) \leftrightarrow \alpha^{c2*}$ and is computed using the Fourier quantifier elimination. The propagated-constraint section $\beta_i^{p*} = \alpha^{c1} \wedge \alpha^{c2*} \wedge \alpha^p$.

According to Definition 5.3.1.3 of working formula, since we have $\neg^9$, then the sixth condition of this definition holds, thus $\beta_i^p = \alpha^c \wedge \alpha^p$. The left hand side of this rule is equivalent in T^*_{ad} to a formula of the form

$$\neg(\exists \bar{x}\, \alpha^c \wedge \alpha^p \wedge \bigwedge_i \neg(\alpha^c \wedge \alpha^p \wedge (\exists \bar{y}_i \beta_i^c \wedge \varphi_i))),$$

i.e. to

$$\neg(\exists\bar{x}\,\alpha^c \wedge \alpha^p \wedge \bigwedge_i \neg(\exists\bar{y}_i\beta^c_i \wedge \varphi_i)).$$

which according to Property 5.3.1.6 is equivalent in T^*_{ad} to a formula of the form

$$\neg(\alpha^p \wedge (\exists\bar{x}\,\alpha^c \wedge \bigwedge_i \neg(\exists\bar{y}_i\beta^c_i \wedge \varphi_i))).$$

According to Definition 5.3.1.3 of working formula, since we have $\neg^7$, then the conditions 4,5,6 and 7 of this definition hold. Thus the formula $\exists\bar{x}\alpha^c$ can be decomposed in T^*_{ad} with $\exists\bar{x}^3\alpha^{c3} = \exists\varepsilon\, true$. The preceding formula is thus equivalent in T^*_{ad} to

$$\neg(\alpha^p \wedge (\exists\bar{x}^1\alpha^{c1} \wedge (\exists\bar{x}^2\alpha^{c2} \wedge \bigwedge_i \neg(\exists\bar{y}_i\beta^c_i \wedge \varphi_i)))).$$

Let us denote by I_1, the set of the $i \in I$ such that x^2_n has no occurrences in $\exists\bar{y}^1_i\beta^{c1}_i$. The preceding formula is equivalent in T^*_{ad} to

$$\neg(\alpha^p \wedge (\exists\bar{x}^1\alpha^{c1} \wedge \alpha^p \wedge \begin{bmatrix}\exists x^2_1 ... \exists x^2_{n-1}\\ (\bigwedge_{i\in I_1} \neg(\exists\bar{y}^1_i\beta^{c1}_i))\wedge\\ (\exists x^2_n\,\alpha^{c2} \wedge \bigwedge_{i\in I-I_1} \neg(\exists\bar{y}^{c1}_i\beta^{c1}_i))\end{bmatrix})). \tag{5.14}$$

According to the properties 3.1.0.4, 5.2.3.2 and 5.2.3.3, the formula (5.14) is equivalent in T^*_{ad} to

$$\neg(\alpha^p \wedge (\exists\bar{x}^1\alpha^1 \wedge \alpha^p \wedge \begin{bmatrix}\exists x^2_1 ... \exists x^2_{n-1}\\ (\bigwedge_{i\in I_1} \neg(\exists\bar{y}^1_i\beta^{c1}_i))\wedge\\ (\exists x^2_n\,\alpha^{c2})\end{bmatrix})). \tag{5.15}$$

According to Property 5.2.3.4, there exists $\alpha^{c2}_n \in A^2$ such that $T^*_{ad} \models (\exists x^2_n\,\alpha^{c2}) \leftrightarrow \alpha^{c2}_n$ with $\alpha^{c2}_n \in A^2$. The preceding formula is thus equivalent in T^*_{ad} to

$$\neg(\alpha^p \wedge (\exists\bar{x}^1\alpha^{c1} \wedge \alpha^p \wedge (\exists x^2_1 ... \exists x^2_{n-1} \bigwedge_{i\in I_1} \neg(\exists\bar{y}^1_i\beta^{c1}_i) \wedge \alpha^{c2}_n))),$$

i.e. to

$$\neg(\alpha^p \wedge (\exists\bar{x}^1\alpha^{c1} \wedge \alpha^p \wedge (\exists x^2_1 ... \exists x^2_{n-1}\,\alpha^{c2}_n \wedge \bigwedge_{i\in I_1} \neg(\exists\bar{y}^1_i\beta^{c1}_i)))).$$

By repeating the preceding steps $(n-1)$ time and by denoting by I_k the sets of the $i \in I_{k-1}$ such that $x^2_{(n-k+1)}$ has no occurrences in $\exists\bar{y}^1_i\beta^{c1}_i$, the preceding formula is equivalent in T^*_{ad} to

$$\neg(\alpha^p \wedge (\exists\bar{x}^1\alpha^{c1} \wedge \alpha^{c2}_1 \wedge \alpha^p \wedge \textstyle\bigwedge_{i\in I_n} \neg(\exists\bar{y}^1_i\beta^{c1}_i))),$$

i.e. to

$$\neg(\exists\bar{x}^1\alpha^{c1} \wedge \alpha^{c2}_1 \wedge \alpha^p \wedge \bigwedge_{i\in I_n} \neg(\exists\bar{y}^1_i\beta^{c1}_i \wedge \alpha^{c1} \wedge \alpha^{c2}_1 \wedge \alpha^p)).$$

This rule is thus correct in T^*_{ad}.

Correctness of the rule 26

$$\neg^7\begin{bmatrix}\exists\bar{x}\,\alpha^c \wedge \alpha^p \wedge \varphi\wedge\\ \neg^8(\exists\bar{y}\beta^c \wedge \beta^p)\end{bmatrix} \Longrightarrow \begin{bmatrix}\neg^7(\exists\bar{x}\,\alpha^c \wedge \alpha^p \wedge \varphi \wedge \neg^9(\exists\bar{y}\beta^{c1} \wedge \beta^p))\wedge\\ \bigwedge_{i\in I} \neg^1(\exists\bar{x}\bar{y}\beta^p \wedge \beta^{c1} \wedge \beta^{c2*}_i \wedge \varphi_0)\end{bmatrix}$$

where φ is such that every negation symbol $\neg^k$ has $k \geq 6$, φ_0 is obtained from φ by replacing all occurrences of $\neg^k$ by $\neg^0$ and all propagated-constraint sections by *true*. Let β^2 be the formula obtained from β^{c2} by removing the multiple occurrences of typing constraints and for all the variables y which do not occur in an inequation of β^{c2} we remove all relation *numy* or $\neg$*numy* which are both in β^{c1} and β^{c2}. If β^2 is the formula *true* then $I = \emptyset$, otherwise the β_i^{c2*} with $i \in I$ are obtained from β^2 as follows: Since $\beta^2 \in A^2$ then it is of the form

$$\begin{bmatrix}(\bigwedge_{\ell\in L} num z_\ell) \wedge (\bigwedge_{k\in K} \neg num v_k)\wedge \\ ((\bigwedge_{j\in J} \sum_{i=1}^n a_{ij}x_i < a_{0j}) \wedge \bigwedge_{m=1}^n num x_m)\end{bmatrix},$$

thus $\neg\beta^2$ is of the form

$$\begin{bmatrix}(\bigvee_{\ell\in L} \neg num\, z_\ell) \vee (\bigvee_{k\in K} num v_k) \vee (\bigvee_{m=1}^n \neg num x_m)\vee \\ \bigvee_{j\in J}((\sum_{i=1}^n a_{ij}x_i = a_{0j}1 \wedge \bigwedge_{m=1}^n num x_m)\vee \\ (\sum_{i=1}^n (-a_{ij})x_i < (-a_{0j})1 \wedge \bigwedge_{m=1}^n num x_m))\end{bmatrix}$$

Each element of this disjunction is a block and represents a formula β_i^{c2*}. Of course we have $T^*_{ad} \models (\neg\beta^2) \leftrightarrow \bigvee_i \beta_i^{c2*}$.

Since we have $\neg^8$, then according to Definition 5.3.1.3, the formula $\exists\bar{y}\beta^c$ is equivalent in T^*_{ad} to a decomposed formula of the form $\exists\bar{y}\beta^{c1} \wedge \beta^{c2}$ with $\exists\bar{y}\beta^{c1} \in A^1$. Let β^2 be the formula obtained from β^{c2} by removing from β^{c2} the multiple occurrences of the typing constraints and by removing from β^{c2} all the relations *num* y or *tree* y which are both in β^{c1} and β^{c2} for every variable y which has no occurrences in the inequations of β^{c2}. The left hand side of this rule is equivalent in T^*_{ad} to

$$\neg(\exists\bar{x}\alpha^c \wedge \alpha^p \wedge \varphi \wedge \neg(\exists\bar{y}\beta^p \wedge \beta^{c1} \wedge \beta^2)).$$

According to the definition of the set A^1 and Property 5.2.1.1, we have $T^*_{ad} \models \exists?\bar{y}\beta^{c1}$, thus $T^*_{ad} \models \exists?\bar{y}\beta^{c1} \wedge \beta^p$. According to Property 3.1.0.6 of Chapter 3, the preceding formula is equivalent in T^*_{ad} to

$$\neg(\exists\bar{x}\alpha^c \wedge \alpha^p \wedge \varphi \wedge \neg(\exists\bar{y}\beta^p \wedge \beta^{c1})) \vee \neg(\exists\bar{x}\bar{y}\beta^p \wedge \beta^{c1} \wedge \neg\beta^2 \wedge \varphi).$$

According to the conditions of this rule, the formula $\neg\beta^2$ is equivalent in T^*_{ad} to a disjunction of the form $\bigvee_{i\in I} \beta_i^{c2*}$. The preceding formula is thus equivalent in T^*_{ad} to

$$\neg(\exists\bar{x}\alpha^c \wedge \alpha^p \wedge \varphi \wedge \neg(\exists\bar{y}\beta^p \wedge \beta^{c1})) \wedge \bigwedge_{i\in I} \neg(\exists\bar{x}\bar{y}\beta^p \wedge \beta^{c1} \wedge \beta_i^{c2*} \wedge \varphi).$$

Thus this rule is correct in T^*_{ad}.

Correctness of the rule 27

$$\neg^7(\exists\bar{x}\,\alpha^c \wedge \alpha^p \wedge \varphi \wedge \neg^9(\exists\varepsilon\, true \wedge \beta^p)) \Longrightarrow true.$$

According to the properties of $\neg^9$ in Definition 5.3.1.3, the formula β^p is the formula $\alpha^c \wedge \alpha^p$. This rule is thus correct in T^*_{ad}.

Correctness of the rule 28

$$\neg^7\begin{bmatrix}\exists\bar{x}\,\alpha^c \wedge \alpha^p \wedge \varphi\wedge \\ \neg^8\begin{bmatrix}\exists\bar{y}\,\beta^c \wedge \beta^p\wedge \\ \bigwedge_{i\in I}\neg^9(\exists\bar{z}_i\,\gamma_i^c \wedge \gamma_i^p)\end{bmatrix}\end{bmatrix} \Longrightarrow \begin{bmatrix}\neg^7(\exists\bar{x}\,\alpha^c \wedge \alpha^p \wedge \varphi \wedge \neg^8(\exists\bar{y}\,\beta^c \wedge \beta^p))\wedge \\ \bigwedge_{i\in I}\neg^6(\exists\bar{x}\bar{y}\bar{z}_i\,\delta_i^c \wedge \delta_i^p \wedge \varphi_0)\end{bmatrix}$$

with $I \neq \emptyset$, φ is such that every negation symbol $\neg^k$ has $k \geq 6$, φ_0 is obtained from φ by replacing all occurrences of $\neg^k$ by $\neg^0$ and all propagated-constraint sections by *true*. The formula $\delta_i^p = \alpha^p$, $\delta_i^c = \gamma_i^c \wedge \beta^c \wedge \alpha^c$.

According to the properties of $\neg^8$ and $\neg^9$ in Definition 5.3.1.3, we have $\beta^p = \alpha^c \wedge \alpha^p$ and $\gamma^p = \alpha^c \wedge \alpha^p \wedge \beta^c$. The left hand side of this rule is equivalent in T^*_{ad} to

$$\neg(\exists \bar{x}\alpha^c \wedge \alpha^p \wedge \varphi \wedge \neg(\exists \bar{y}\beta^c \wedge \bigwedge_{i\in I} \neg(\exists \bar{z}_i \gamma_i^c))). \tag{5.16}$$

According to the properties of $\neg^9$ and the definition of the set A^1, all the variables of $\bar{y}$ are reachable in $\exists \bar{y}\beta^c$. Thus, according to Property 5.2.1.1, we get $T^*_{ad} \models \exists ? \bar{y}\beta^c$. According to Corollary 3.1.0.6 defined in Chapter 3, the formula (5.16) is equivalent in T^*_{ad} to

$$\neg((\exists \bar{x}\alpha^c \wedge \alpha^p \wedge \varphi \wedge \neg(\exists \bar{y}\beta^c)) \vee \bigvee_{i\in I} (\exists \bar{x}\bar{y}\bar{z}_i \gamma_i^c \wedge \beta^c \wedge \alpha^c \wedge \alpha^p \wedge \varphi)),$$

i.e. to

$$\neg(\exists \bar{x}\alpha^c \wedge \alpha^p \wedge \varphi \wedge \neg(\exists \bar{y}\beta^c \wedge \alpha^c \wedge \alpha^p)) \wedge \textstyle\bigwedge_{i\in I} \neg(\exists \bar{x}\bar{y}\bar{z}_i \gamma_i^c \wedge \beta^c \wedge \alpha^c \wedge \alpha^p \wedge \varphi).$$

Since $\beta^p = \alpha^c \wedge \alpha^p$, $\delta_i^p = \alpha^p$ and $\delta_i^c = \gamma_i^c \wedge \beta^c \wedge \alpha^c$ then the preceding formula is equivalent in T^*_{ad} to

$$\neg^7(\exists \bar{x}\, \alpha^c \wedge \alpha^p \wedge \varphi \wedge \neg^8(\exists \bar{y}\, \beta^c \wedge \beta^p)) \wedge \bigwedge_{i\in I} \neg^6(\exists \bar{x}\bar{y}\bar{z}_i\, \delta_i^c \wedge \delta_i^p \wedge \varphi_0).$$

Thus this rule is correct in T^*_{ad}.

Proof third part: Let us show that every finite application of the rules on an initial working formula produces a final working formula. Let φ be an initial working formula of the form $\neg^6(\exists \varepsilon\, true \wedge \bigwedge_{i\in I} \varphi_i)$, where all the negations of the φ_i are of the form $\neg^0$. The only rule which can be applied is the rule 24, which starts the solving process by a top-down simplification and propagation of constraints. It is the rules 1...24 which will be used in these steps. At the end, all the sub-working formulas contain negations of the form $\neg^7$. The rule 25

$$\neg^7 \begin{bmatrix} \exists \bar{x}\, \alpha^c \wedge \alpha^p \wedge \\ \bigwedge_{i\in I} \neg^9(\exists \bar{y}_i \beta_i^c \wedge \beta_i^p) \end{bmatrix} \Longrightarrow \neg^8 \begin{bmatrix} \exists \bar{x}^1 \alpha^{c1} \wedge \alpha^{c2*} \wedge \alpha^p \wedge \\ \bigwedge_{i\in I'} \neg^9(\exists \bar{y}_i \beta_i^c \wedge \beta_i^{p*}) \end{bmatrix}$$

with $I = \emptyset$ can now be applied on the most nested sub-working formulas and changes the negations from $\neg^7$ into $\neg^8$, then the rule 26,

$$\neg^7 \begin{bmatrix} \exists \bar{x}\, \alpha^c \wedge \alpha^p \wedge \varphi \wedge \\ \neg^8(\exists \bar{y}\beta^c \wedge \beta^p) \end{bmatrix} \Longrightarrow \begin{bmatrix} \neg^7(\exists \bar{x}\, \alpha^c \wedge \alpha^p \wedge \varphi \wedge \neg^9(\exists \bar{y}\beta^{c1} \wedge \beta^p)) \wedge \\ \bigwedge_{i\in I} \neg^1(\exists \bar{x}\bar{y}\beta^p \wedge \beta^{c1} \wedge \beta_i^{c2*} \wedge \varphi_0) \end{bmatrix}$$

can be applied and changes every sequence of the form $\neg^7\neg^8$ into a sequence of the form $\neg^7\neg^9$. This rule creates also a conjunction of working formulas each one containing negations of the form $\neg^1$, on which the first steps will be applied again. When a sequence $\neg^7\neg^9$ is obtained, then the rule 27

$$\neg^7(\exists \bar{x}\, \alpha^c \wedge \alpha^p \wedge \varphi \wedge \neg^9(\exists \varepsilon true \wedge \beta^p)) \Longrightarrow true$$

or the rule 25 can be applied and changes the internal negations from $\neg^7\neg^8$ to $\neg^8\neg^9$. When we have only sequences of the form $\neg^7\neg^8\neg^9$ we can decrease the depth of the working formula from 3 to 2 by applying the rule 28

$$\neg^7 \begin{bmatrix} \exists \bar{x}\, \alpha^c \wedge \alpha^p \wedge \varphi \wedge \\ \neg^8 \begin{bmatrix} \exists \bar{y}\, \beta^c \wedge \beta^p \wedge \\ \bigwedge_{i\in I} \neg^9(\exists \bar{z}_i\, \gamma_i^c \wedge \gamma_i^p) \end{bmatrix} \end{bmatrix} \Longrightarrow \begin{bmatrix} \neg^7(\exists \bar{x}\, \alpha^c \wedge \alpha^p \wedge \varphi \wedge \neg^8(\exists \bar{y}\, \beta^c \wedge \beta^p)) \wedge \\ \bigwedge_{i\in I} \neg^6(\exists \bar{x}\bar{y}\bar{z}_i\, \delta_i^c \wedge \delta_i^p \wedge \varphi_0) \end{bmatrix}$$

All these steps are repeated until reaching the following formula

$$\neg^7(\exists\varepsilon\, true \wedge \bigwedge_{i\in I} \neg^8(\exists\bar{x}_i\, \alpha_i^c \wedge \alpha_i^p \wedge \bigwedge_{j\in J_i} \neg^9(\exists\bar{y}_{ij}\, \beta_{ij}^c \wedge \beta_{ij}^p))),$$

which is a final working formula.

5.3.4 The algorithm of resolution

Solving a general constraint φ in T^*_{ad} proceeds as follows:

1. Transform φ into a normalized formula, then into an initial working formula ϕ, which is equivalent to φ in T^*_{ad}.
2. Transform ϕ into a final working formula ψ using the rewriting rules defined in the subsection 5.3.3.
3. Extract from ψ the disjunction of general solved formulas, equivalent to ψ in T^*_{ad}. If the disjunction contains the general solved formula *true*, then it is reduced to *true*.

Example 5.3.4.1 *Let φ be the following constraint having i, j as free variables:*

$$\exists x\, x = fij \wedge i > 0 \wedge \neg numx \wedge numi \wedge numj \wedge \neg(\exists k\, j = 2k \wedge numk).$$

We can see that $numj \wedge \neg(\exists k\, j = 2k \wedge numk)$ is always false in T^*_{ad} since for every variable j, there exists a unique variable k such that $j = 2k$ (axiom 13_n). Let us transform φ into an initial working formula (the propagated-constraint sections are underlined):

$$\neg^6\neg^0(\exists x\, x = fij \wedge i > 0 \wedge \neg numx \wedge numj \wedge \underline{true} \wedge \neg^0(\exists k\, j = 2k \wedge numk \wedge \underline{true}))$$

After having applied the rules 24, 15, 16, 15, 19, 21, 22, 23 in this order, we obtain

$$\neg^7\neg^6(\exists x\, x = fij \wedge i > 0 \wedge \neg numx \wedge numi \wedge numj \wedge \underline{true} \wedge \neg^0(\exists k\, j = 2k \wedge numk \wedge \underline{true}))$$

The rule 24 can now be applied, we get

$$\neg^7\neg^7 \left[\begin{array}{l} i > 0 \wedge numi \wedge numj \wedge \underline{true} \wedge \\ \neg^1(\exists xk\, x = fij \wedge j = 2k \wedge numk \wedge \neg numx \wedge \underline{i > 0 \wedge numi \wedge numj}) \end{array} \right]$$

After having applied on the sub-working formula $\neg^1(...)$ the rules 15, 19, 21, 12, 22, 23 the preceding formula is equivalent to

$$\neg^7\neg^7 \left[\begin{array}{l} i > 0 \wedge numi \wedge numj \wedge \underline{true} \wedge \\ \neg^6(\exists xk\, x = fij \wedge j - 2k = 0 \wedge numk \wedge \neg numx \wedge \underline{i > 0 \wedge numi \wedge numj}) \end{array} \right]$$

The rule 24 can be applied. We get

$$\neg^7\neg^7(i > 0 \wedge numi \wedge numj \wedge \underline{true} \wedge \neg^7(true \wedge \underline{i > 0 \wedge numi \wedge numj}))$$

The rules 25, 26 are applied in this order and we obtain

$$\neg^7\neg^7(i > 0 \wedge numi \wedge numj \wedge \underline{true} \wedge \neg^9(true \wedge \underline{i > 0 \wedge numi \wedge numj}))$$

Finally, by application of the rule 27, we get the final working formula $\neg^7 true$, which is equivalent to the empty disjunction of general solved formulas, i.e. false.

Since T^*_{ad} has at least one model and according to properties 5.3.3.1, 5.3.1.5 and 5.3.1.7, we get the following corollary

Corollary 5.3.4.2 *Each formula is equivalent in T^*_{ad}, either to true, or to false, or to a disjunction of general solved formulas having at least one free variable and being equivalent neither to true nor to false in T^*_{ad}.*

Chapter 6

Conclusion

We have presented in this thesis new classes of theories and given for each one a decision procedure. We have also presented an automatic way to combine any first order theory T with the theory of finite or infinite and have shown that if T is flexible than T^* is complete. We have ended this thesis by a general algorithm solving any first order constraint in a combination of trees and rational numbers.

S. Vorobyov [41] have shown that the problem of deciding if a proposition is true or not in the theory of finite or infinite trees is non-elementary, i.e. the complexity of all algorithms solving propositions is not bounded by a tower of powers of $2's$ (top down evaluation) with a fixed height. A. Colmerauer and T. Dao [7] have also given a proof of non-elementary complexity of solving constraints in this theory. As a consequence, the complexity of our algorithm and the size of our solved formulas are of this order. We can show easily that the size of our solved formulas is bounded above by a top down tower of powers of $2's$, whose height is the maximal depth of nested negations in the initial formula. The function $\alpha(\varphi)$ used to show the termination of our rules illustrates this result. However, the general algorithm of Chapter 5 is reduced to the algorithm of [16] if the initial formula contains only tree constraints well typed. In this case, we will get without any doubts the same performances than those of [16], i.e. solving formulas having until 160 nested alternated quantifiers ($\exists\forall$). On the other hand, the constraints expressing k-winning positions in [16, 7] can be expressed in a much more easy way in an extension into trees of positive integer numbers. In fact, while the integer a is expressed in [16] by the tree [25] $f^a(0)$, this integer will be expressed directly using the term a in the extension into trees of positive integer numbers. This simplification will probably enable us to get better performances in terms of time-execution and maximal depth of solved formulas comparing with those of [16].

Currently, we try to find a more abstract characterization and/or a model theoretic characterization of the decomposable theories. The actual definition gives only algorithmic insight into what it means for a theory to be complete. We expect to add new vectorial quantifiers in the decomposition such as $\exists^n$ which means *there exists* n and $\exists^{\Psi(u)}_{n,\infty}$ which means *there exists* n *or infinite*, in order to increase the size of the set of decomposable theories and may be get an abstract definition much more simple than the one defined in this thesis. Another interesting challenge is to find which special quantifiers must be added to the decomposable theories to get an equivalence between complete theory and decomposable theory. A first attempt on this subject is actually in progress using the quantifiers $\exists^n$ and $\exists^{\Psi(u)}_{n,\infty}$. It would be also interesting to show if these new quantifiers are enough to prove that every theory which accepts elimination of quantifiers is decomposable.

[25] Of course $f^0(x) = x$ and $f^{a+1}(x) = f(f^a(x))$

We have also established a long list of infinite and zero-infinite decomposable theories. We can cite for example: theory of finite trees, theory of infinite trees, theory of finite or infinite trees [19], theory of additive rational or real numbers with addition and subtraction, theory of linear dense order without endpoints, theory of ordered additive rational or real numbers with addition, subtraction and a linear dense order relation without endpoints, combination of tress and ordered additive rational numbers [24], construction of trees on an ordered set [23], extension of trees by first-order theories [20].

Currently, we are showing the decomposability of other fundamental theories such as: theory of lists using a combination of particular trees, theory of queues as it has done in [39], and the combination of trees and real numbers together with addition, subtraction, multiplication and a linear dense order relation without endpoints. We try also to find some formal methods to get easily the sets $\psi(u)$, A', A'' and A''' for any decomposable theory T.

Our initial aim in this thesis was to give axiomatizations of complex theories around trees and show their completeness. We have made better by introducing the term *extension into trees of theories* and by giving conditions on T and only on T so that the theory T+trees is complete. We have also shown the completeness of a theory built on the model of Prolog III, which was unproved before. In order to extend this theoretical work, we plan with Thom Fruehwirth [27] to add to CHR a general mechanism to treat our normalized formulas. This will enable us to implement quickly and easily our algorithms and get a general idea on the expressiveness of first order constraints in complete theories around trees.

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Abstract

The goal of this thesis is the study of a harmonious way to combine any first order theory with the theory of finite or infinite trees. For that:

First of all, we introduce two classes of theories that we call *infinite-decomposable* and *zero-infinite-decomposable*. We show that these theories are complete and accept a decision procedure which for every proposition gives either *true* or *false*. We show also that these classes of theories contain a large number of fundamental theories used in computer science, we can cite for example: the theory of additive rational or real numbers, the theory of the linear dense order without endpoints, the theory of finite or infinite trees, the construction of trees on an ordered set, and a combination of trees and ordered additive rational or real numbers.

We give then an automatic way to combine any first order theory T with the theory of finite or infinite trees. A such hybrid theory is called *extension into trees* of the theory T and is denoted by T^*. After having defined the axiomatization of T^* using those of T, we define a new class of theories that we call *flexible* and show that if T is flexible then T^* is zero-infinite-decomposable and thus complete. The flexible theories are first order theories having elegant properties which enable us to handle easily first order formulas. We show among other theories that the theory T_{ad} of ordered additive rational numbers is flexible and thus that the extension into trees T^*_{ad} of T_{ad} is complete.

Finally, we end this thesis by a general algorithm for solving efficiently first order constraints in T^*_{ad}. The algorithm is given in the form of 28 rewriting rules which transform every formula φ, which can possibly contain free variables, into a disjunction ϕ of solved formulas equivalent to φ in T^*_{ad} and such that ϕ is either the formula *true*, or the formula *false*, or a formula having at least one free variable and being equivalent neither to *true* nor to *false* in T^*_{ad}. Moreover, the solutions of the free variables of ϕ are expressed in a clear and explicit way in ϕ.

Keywords: Theory of finite or infinite trees, Complete theory, Combination of theories, Solving first order constraints, Rewriting rules.